PROBLEMS, MATH 215A

Do 28 problems, due Friday Dec 2.
All rings are commutative.

1. Prove that if x is invertible in a ring A and $y \in A$ is nilpotent, then $x + y \in A^{\times}$.

2. Let $f = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0 \in A[X]$ be a polynomial. Prove that $f \in A[X]^{\times}$ if and only if $a_0 \in A^{\times}$ and a_i are nilpotent for all $i \geq 1$.

3. Prove that for every nonzero ring A, the set $\mathrm{Spec}(A)$ has a minimal element with respect to inclusion.

4. Let $f : A \to B$ be a ring homomorphism, $\mathfrak{b} \subset B$ an ideal. Prove that the closure of $f^*(V(\mathfrak{b}))$ in $\mathrm{Spec}(A)$ coincides with $V(\mathfrak{b}^c)$.

5. Let M and M be finitely generated modules over a local ring A. Prove that is $M \otimes_A N = 0$, then $M = 0$ or $N = 0$.

6. Prove that if B is a flat A-algebra and N is a flat B-module, then N is a flat A-module.

7. Prove that every module is a colimit of free modules.

8. Prove that if $A^n \to A^m$ is a surjective A-module homomorphism, then $n \geq m$.

9*. Prove that if $A^n \to A^m$ is an injective A-module homomorphism, then $n \leq m$.

10. Let $0 \to M \to N \to P \to 0$ be an exact sequence of modules over A and P is flat A-module, then for every module X, the sequence $0 \to M \otimes_A X \to N \otimes_A X \to P \otimes_A X \to 0$ is exact.

11. Let $f : M \to M$ be a surjective endomorphism of a finitely generated A-module. Prove that $f^n + a_1 f^{n-1} + \cdots + a_1 f + a_n = 0$ for some n and $a_1, \ldots, a_n \in A$ with $a_n \neq 0$.

12*. Let $f : M \to M$ be a surjective endomorphism of a finitely generated A-module. Prove that f is an isomorphism.

13. Let M be an A-module and $\mathfrak{a} \subset A$ an ideal. Suppose $M_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$. Prove that $M = \mathfrak{a}M$.

14. Let $A \to B \to C$ be ring homomorphisms. Prove that if C is flat over A and C is faithfully flat over B, then B is flat over A.

15. Let M be a finitely presented A-module. Prove that for every surjective homomorphism $f : X \to M$ with finitely generated X, the kernel of f is also finitely generated.

16. Prove that the functor lim is left-exact. Precisely, let $0 \to M \to N \to P$ be an exact sequence of functors $I \to A - Mod$, i.e., for every i in I the

sequence of A-modules $0 \to M_i \to N_i \to P_i$ is exact. Show that the sequence $0 \to \lim M \to \lim N \to \lim P$ is exact. Prove the functor colim is right-exact.

17. A nonempty category I is called *filtered* if for every two objects i and i' there are morphisms $i \to j$ and $i' \to j$ for some object j and for every two morphisms $f : i \to j$ and $f' : i \to j'$, there exist morphisms $g : j \to k$ and $g' : j' \to k$ for some object k such that $gf = g'f'$. Prove that the colimit over a filtered category is an exact functor.

18. Let M be an A-module and let $(a_1, a_2, \dots, a_n) \in A^n$ be a unimodular row. Suppose we are given elements $m_i \in M_{a_i}$ for every i such that for every i and j, the images of m_i and m_j in $M_{a_i a_j}$ coincide. Prove that there is a unique element $m \in M$ such that the image of m in M_{a_i} is equal to m_i for every i.

19. Let M and N be A-modules and let $(a_1, a_2, \dots, a_n) \in A^n$ be a unimodular row. Suppose we are given A_{a_i}-module homomorphisms $\varphi_i : M_{a_i} \to N_{a_i}$ for every i such that for every i and j, the images of φ_i and φ_j in $\mathrm{Hom}_{A_{a_i a_j}}(M_{a_i a_j}, N_{a_i a_j})$ coincide. Prove that there is a unique homomorphism of A-modules $\varphi : M \to N$ such that the image of φ in $\mathrm{Hom}_{A_{a_i}}(M_{a_i}, N_{a_i})$ is equal to φ_i for every i.

20. Let P be a finitely generated projective A-module. Prove that there is a Noetherian subring A_0 of A and a projective A_0-module P_0 such that $P \simeq P_0 \otimes_{A_0} A$.

21*. Let P and Q be A-modules such that $P \otimes_A Q \simeq A$. Prove that P and Q are finitely generated projective modules of constant rank 1.

22. Let M be an A-module. The *exterior n-th power $\Lambda^n(M)$ of M* is the factor module of the tensor product of n copies of M by the submodule generated by the tensors $m_1 \otimes m_2 \otimes \cdots \otimes m_n$ such that $m_i = m_j$ for some $i \neq j$. Prove that is P is a projective A-module of constant rank n, then $\Lambda^n(M)$ is a projective module of constant rank 1.

23. Let P and Q be two projective modules of constant rank 1. Prove that if P and Q are stably isomorphic, then $P \simeq Q$.

24. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a ring A such that $\mathrm{Spec}(A)$ is the disjoint union of the closed sets $V(\mathfrak{a})$ and $V(\mathfrak{b})$. Prove that there is an idempotent $e \in A$ such that $V(\mathfrak{a}) = V(Ae)$ and $V(\mathfrak{b}) = V(A(1-e))$.

25. Prove that for every $n > 0$, there is a ring A_n and a unimodular n-row a over A_n such that for every ring B and a unimodular n-row b over B there exists a ring homomorphism $f : A_n \to B$ with $f(a) = b$.

26. Let $x, y \in A$ be two elements generating the unit ideal and $\varphi \in GL_n(A_{xy})$. Prove that the A-module

$$P_\varphi := \{(u, v) \in A_x \oplus A_y \quad \text{such that} \quad \varphi(u_y) = v_x \quad \text{in} \quad A^n_{xy}\}$$

is projective with $P_x \simeq A^n_x$ and $P_y \simeq A^n_y$.

27*. Let $n \geq m$ be positive integers. Prove that there exists a ring $A_{n,m}$ and a projective $A_{n,m}$-module $P_{n,m}$ of constant rank m generated by n elements such for every ring A and every projective A-module P of constant rank m

generated by n elements, there is a ring homomorphism $f : A_{n,m} \to A$ with $P \simeq P_{n,m} \otimes_{A_{n,m}} A$.

28. Let P be a finitely generated projective A-module. Prove that A is the product $A_1 \times A_2 \times \cdots \times A_n$ of rings and $P = P_1 \times P_2 \times \cdots \times P_n$, where P_i is finitely generated projective A_i-module of constant rank for every i.

29. Let P be an A-module and $n \geq 0$ an integer. Prove that TFAE:
(a) P is a finitely generated projective A-module of constant rank n;
(b) P is finitely generated and $P_{\mathfrak{m}}$ is free of rank n for all $\mathfrak{m} \in Max(A)$;
(c) P is finitely generated and $P_{\mathfrak{p}}$ is free of rank n for all $\mathfrak{p} \in Spec(A)$;
(d) There is a unimodular row $(a_1, a_2, \ldots, a_n)$ such that P_{a_i} is a free A_{a_i}-module of rank n for all i.

30. Let P be a finitely generated projective A-module of constant rank 1. Prove that $\mathrm{End}_A(P) \simeq A$.

31. Let P be a finitely generated projective A-module. Prove that P is faithfully flat if and only if $aP \neq 0$ for every $0 \neq a \in A$.

32. Let $A = A_0 \oplus A_1 \oplus \ldots$ be a graded commutative ring. Prove that A is Noetherian if and only if A_0 is Noetherian and the A_0-algebra A is finitely generated.

33. Let R be a local ring, $f \in R[t_1, \ldots, t_n]$ a polynomial. Suppose that one of the coefficients of f is invertible in R. Prove that f is a non-zero-divisor in $R[t_1, \ldots, t_n]$.

34. Let R be a complete Noetherian local ring with maximal ideal P. Suppose there is a subfield $K \subset R$ mapping isomorphically onto R/P. Let $x_1, \ldots, x_d$ be a system of parameters of R. Prove that R is a finitely generated module over the subring $K[[x_1, \ldots, x_d]]$.

35. Let f and g be functions $\{0, 1, 2, \ldots\} \to \mathbb{Z}$ such that $f(n+1) - f(n) = g(n)$ for all $n \geq 0$. Prove that if g is a polynomial of degree d then f is a polynomial of degree $d + 1$.

36. Let R be a Noetherian local ring with maximal ideal P and $x \in P$. Prove that

$$\dim R \geq \dim(R/xR) \geq \dim R - 1.$$

37. Let R be a Noetherian local ring with maximal ideal P and $a_1, a_2, \ldots, a_k \in P$. Prove that $\{a_1, a_2, \ldots, a_k\}$ is a part of a system of parameters if and only if $\dim\big(R/\langle a_1, a_2, \ldots, a_k\rangle\big) = \dim R - k$.

38. Let $R \to S$ be a flat local homomorphism of Noetherian local rings, P the maximal ideal of R. Prove that is R and S/PS are regular then so is S.