PROBLEMS, MATH 214B

1. Let P be a module over a commutative ring A. Prove that P is a finitely generated projective A-module if and only if there are elements $f_1, f_2, \dots, f_n$ in A generating the unit ideal and such that P_{f_i} is a free A_{f_i}-module of finite rank for every i.

2. Let V be a finite dimensional vector space over a field K. Prove that every section of the tautological line bundle L_t over $\mathbb{P}_k(V)$ is trivial.

3. Let k be a field. The projective line $\mathbb{P}^1_k$ is covered by two open sets U_1 and U_2, both isomorphic to $\mathbb{A}^1_k$, in a standard way. The intersection $U_1 \cap U_2$ is $\operatorname{Spec} k[t, t^{-1}]$. Let $\alpha \in \mathrm{GL}_n(k[t, t^{-1}])$ for some n. Write E_α for the vector bundle over $\mathbb{P}^1_k$ which is obtained by gluing the trivial rank n vector bundles over U_1 and U_2 along the isomorphism over $U_1 \cap U_2$ given by the matrix α. Prove that $E_\alpha \simeq E_{\alpha'}$ if and only if there are $\beta \in \mathrm{GL}_n(k[t])$ and $\gamma \in \mathrm{GL}_n(k[t^{-1}])$ such that $\alpha' = \beta\alpha\gamma$.

4. Prove that every vector bundle over $\mathbb{P}^1_k$, k a field, is isomorphic to a direct sum of tensor powers of the tautological line bundle.

5. Classify line bundles over the affine line with double origin.

6. Let $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n$ be line bundles over a scheme X. Prove that

$$\Lambda^n(\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \dots \oplus \mathcal{L}_n) \simeq \mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \dots \otimes \mathcal{L}_n.$$

7. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be line bundles over a scheme X such that $\mathcal{L}_1 \oplus \mathbb{A}^n_X \simeq \mathcal{L}_2 \oplus \mathbb{A}^n_X$ for some n. Prove that $\mathcal{L}_1 \simeq \mathcal{L}_2$.

8. Let $E \to X$ be a vector bundle, $s : X \to E$ a section. Show that s is a closed embedding.

9. Let $L \to X$ be a line bundle, $z : X \to L$ the zero section. Prove that the bundle is trivial if and only if there is a section $s : X \to L$ such that $s(X) \cap z(X) = \emptyset$.

10. Let E and E' be two vector bundles over $X = \operatorname{Spec} A$. Prove that there is a scheme $I = Iso(E', E)$ over X such that for every commutative A-algebra R the set of R-points of I is the set of isomorphisms between $E' \times_X \operatorname{Spec} R$ and $E \times_X \operatorname{Spec} R$.

11. Prove that a morphism of vector bundles $f : E \to F$ is an admissible monomorphism (i.e., E is a subbundle of F) if and only if f is a closed embedding.

12. Let $E \to X$ be a vector bundle. Prove that there is a line bundle morphism $f : \mathbb{P}_X(E \oplus \mathbb{1}) \setminus \mathbb{P}_X(\mathbb{1}) \to \mathbb{P}_X(E)$ dual to the tautological line bundle over $\mathbb{P}_X(E)$ (thus f is a canonical line bundle over $\mathbb{P}_X(E)$).

13. Let $E \to X$ be a vector bundle, where X is a scheme over K. Determine the set $\mathbb{P}_X(E)(R)$ for a commutative K-algebra R.

14. For an integer $n \geq 2$, let $F_n : K\text{-}Alg \to Sets$ be the functor

$$F_n(R) = U_n(R)/R^\times,$$

where $U_n(R)$ is the set of unimodular n-rows over R. Prove that the functor F_n admits an open cover by affine schemes, but F_n is not local.

15. A subfunctor $F' \subset F$ is called closed if for every morphism $\operatorname{Spec} R \to F$ the morphism $\operatorname{Spec} R \times_F F' \to \operatorname{Spec} R$ is a closed embedding of schemes. Prove that if X' is a closed subscheme of a scheme X, then X' is a closed subfunctor of the functor X.

16. a) Let R be a commutative K-algebra. Let $p_i : R^n \to R$ and $q_j : R^m \to R$ be the projections and $s_{ij} : R^n \otimes R^m \to R$ defined by $s_{ij}(x \otimes y) = p_i(x)q_j(y)$. Prove that a direct summand $M \subset R^n \otimes R^m$ of rank 1 is of the form $M = P \otimes Q$, where P and Q are direct summands of R^n and R^m of rank 1, respectively, if and only if $s_{ij}(m)s_{kl}(m) = s_{il}(m)s_{kj}(m)$ for all $m \in M$ and all i, j, k, l.

b) Prove that the Segre morphism $\mathbb{P}_K^{n-1} \times_K \mathbb{P}_K^{m-1} \to \mathbb{P}_K^{nm-1}$ taking a pair (P, Q) of submodules to $P \otimes Q$ is a closed embedding.

17. a) Let R be a commutative algebra over a field K and V a vector space of dimension n over K. Prove that every right ideal I of $\operatorname{End}_R(V \otimes_K R)$ that is a direct summand of rank n as an R-submodule is of the form $I = \operatorname{Hom}_R(V \otimes_K R, P)$ for a unique direct summand P of $V \otimes_K R$ of rank 1.

b) Prove that the Severi-Brauer variety of the K-algebra $\operatorname{End}_K(V)$ is isomorphic to the projective space $\mathbb{P}_K(V)$.

18. Prove that a local functor covered by open subschemes (not necessarily affine) is a scheme.

19. Let $f \in K$. Prove that the functor $F : K\text{-}Alg \to Sets$ defined by

$$F(R) = \begin{cases} R, & \text{if } f \in R^\times; \\ \emptyset, & \text{otherwise,} \end{cases}$$

is an affine scheme.

20. Let $K \to K'$ be a commutative ring homomorphism and $F : K\text{-}Alg \to Sets$ a functor. Define a functor $F' : K'\text{-}Alg \to Sets$ by $F'(R) = F(R)$. Prove that if F is a scheme, then so is F'.

21. Let X be an integral variety over a field K, $\xi \in X$ the generic point and $L = K(X)$.

a) Let F be a sheaf of sets on X such that for every nonempty open subset $U \subset X$ the natural map $F(U) \to F_\xi$ to the generic stalk is injective. Prove that for every point $x \in X$ the induced map $F_x \to F_\xi$ is injective and $F(X) = \cap_{x \in X} F_x$ in F_ξ.

b) Let Y be a variety over K. Prove that the map $\mathrm{Mor}_K(X,Y) \to Y(L)$, induced by the morphism $\operatorname{Spec} L \to X$, is injective. Show that $\mathrm{Mor}_K(X,Y) = \cap_{x\in X} Y(\mathcal{O}_{X,x})$ in $Y(L)$.

22. Prove that the class of flat morphisms is closed under compositions, base changes and local on the target.

23. Prove that the variety $\operatorname{Spec}\big(K[x,y,z]/(z^2-xy)\big)$ is normal.

24. Compute $\mathrm{Pic}(X)$ for $X = \operatorname{Spec}\big(K[x,y]/(y^2-x^3-x^2)\big)$.

25. Compute $\mathrm{CH}_0(X)$ for $X = \operatorname{Spec}\big(K[x,y]/(y^2-x^3-x^2)\big)$.

26. Compute the image of the class of the tautological line bundle under the isomorphism $\mathrm{Pic}(\mathbb{P}^n_K) \to \mathrm{CH}^1(\mathbb{P}^n_K)$.

27. Let W_1 and W_2 be linear subspaces of the same dimension of a finite dimensional vector space V. Prove that the classes $[\mathbb{P}(W_1)]$ and $[\mathbb{P}(W_2)]$ in $\mathrm{CH}(\mathbb{P}(V))$ coincide.

28. Let X be a closed smooth curve in $\mathbb{A}^2_K$. Prove that the tangent bundle of X is trivial.

29. Let X be a locally factorial integral variety of dimension n over K. Construct an exact sequence

$$1 \to \mathcal{O}_X^\times \to \mathcal{K}_X^\times \to \coprod_{x\in X^{(n-1)}} (i_x)_*(\mathbb{Z}) \to 0,$$

where $i_x : \operatorname{Spec} K(x) \to X$ is a natural morphism.

30. Compute $\mathrm{CH}(\mathbb{P}^n_K \times_K \mathbb{P}^m_K)$.

31. Let $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ be a linear automorphism. Prove that the pull-back homomorphism $f^* : \mathrm{CH}(\mathbb{P}^n_K) \to \mathrm{CH}(\mathbb{P}^n_K)$ is the identity.

32. Let B be an A-algebra. Prove that the morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is flat if and only if B is a flat A-algebra.

33. Is the projection $\operatorname{Spec} K[x,y,z]/(z-xy) \to \operatorname{Spec} K[y,z]$ flat?

34. Let X and Y be smooth varieties over K. Prove that $X \times_K Y$ is also smooth over K.

35. Prove that a morphism $f : X \to Y$ of smooth varieties over K yields a morphism of vector bundles $T_X \to f^*(T_Y)$ over X.

36. Let X and Y be smooth varieties over K and let $p : X \times_K Y \to X$ and $q : X \times_K Y \to Y$ be the two projections. Prove that $T_{X\times_K Y} \simeq p^*(T_X) \oplus q^*(T_Y)$.

37. Let K be a field of characteristic 2 and $a, b \in K^\times$ such that $[K(\sqrt{a},\sqrt{b}) : K] = 4$. Prove that the curve $\operatorname{Spec}\big(K[x,y]/(x^2+ay^2+b)\big)$ is regular but not smooth.

38. Let $f : X \to Y$ be a morphism of varieties.

a) Prove that the morphism $i = (1_X, f) : X \to X \times Y$ is a closed embedding.

b) Prove that i is a regular embedding with the normal bundle $f^*(T_Y)$ if Y is a smooth variety.

39. Let $f : X \to Y$ be a flat morphism of varieties over K of relative dimension n such the fiber of f over any point $y \in Y$ is the affine space $\mathbb{A}^n_{K(y)}$. Prove that the pull-back homomorphism $f^* : \mathrm{CH}_i(Y) \to \mathrm{CH}_{i+n}(X)$ is surjective.

40. Let $f : X \to Y$ and $g : Y \to Z$ be regular closed embeddings. Prove that $g \circ f$ is a regular closed embedding and $(g \circ f)^* = f^* \circ g^*$.