PROBLEMS, MATH 214B

1. Let $\mathcal{F}$ be a presheaf on X. Denote by $Tot(\mathcal{F})$ the disjoint union of the stalks $\mathcal{F}_x$ for all $x \in X$ and by $p : Tot(\mathcal{F}) \to X$ the natural projection. Any section $a \in \mathcal{F}(U)$ over an open subset $U \subset X$ and every point $x \in X$ define an element $a_x \in \mathcal{F}_x$ which then can be viewed as an element of $Tot(\mathcal{F})$ with $p(a_x) = x$. Define topology on $Tot(\mathcal{F})$ with the basis given by the subsets $\{a_x, x \in U\} \subset Tot(\mathcal{F})$ for all open $U \subset X$ and all sections $a \in \mathcal{F}(U)$. A *section* of p over an open set $U \subset X$ is a continuous map $s : U \to Tot(\mathcal{F})$ such that $p(s(x)) = x$ for every $x \in U$. Denote by $\widetilde{\mathcal{F}}$ the sheaf of sections of p. The sheaf $\widetilde{\mathcal{F}}$ is called the *sheaf associated to the presheaf* $\mathcal{F}$.

a) Prove that there is a natural morphism of presheaves $i : \mathcal{F} \to \widetilde{\mathcal{F}}$.

b) Prove that $\mathcal{F}$ is a sheaf iff i is an isomorphism.

c) Prove that for every morphism $j : \mathcal{F} \to \mathcal{G}$ to a sheaf $\mathcal{G}$ there exists a unique morphism $\tilde{j} : \widetilde{\mathcal{F}} \to \mathcal{G}$ such that $\tilde{j} \circ i = j$.

2. Let A be a set, X a topological space. Let $\mathcal{F}$ be the constant presheaf associated with A. Consider the presheaf $\mathcal{G}$ with $\mathcal{G}(U)$ being the set of continuous maps $U \to A$ (we view A as a discrete topological space).

a) Prove that $\mathcal{G}$ is a sheaf.

b) Prove that the natural morphism $i : \mathcal{F} \to \mathcal{G}$ is isomorphic to $i : \mathcal{F} \to \widetilde{\mathcal{F}}$ considered in 1a).

3. Describe $\operatorname{Spec} \mathbb{Z}[X]$.

4. Describe $(\operatorname{Spec} \mathbb{C}) \times_{\operatorname{Spec} \mathbb{R}} (\operatorname{Spec} \mathbb{C})$.

5. Find an example of a commutative ring A that is not Noetherian with Noetherian topological space $\operatorname{Spec} A$.

6. Prove that $\operatorname{Spec} A$ is not connected if and only if A has a nontrivial idempotent (an element $a \neq 0, 1$ such that $a^2 = a$).

7. Let A be a commutative ring, $f \in A$. Prove that the set $\{P \in \operatorname{Spec} A \text{ such that } f(P) = 0\}$ is closed in $\operatorname{Spec} A$.

8. Give an example of a commutative ring A and an open set $U \subset \operatorname{Spec} A$ that is not principal.

9. Prove that the functor $Rings^{op} \to TopSpaces$, $A \mapsto \operatorname{Spec} A$ is neither full nor faithful.

10. Let X be a scheme, $f \in \mathcal{O}_X(X)$.
a) Prove that the set $X_f = \{x \in X \text{ such that } f(x) \neq 0\}$ is open in U.
b) Prove that the restriction of $f \in \mathcal{O}_X(X)$ on the set X_f is invertible.
c) Let $x \in X$, $P_x = \{f \in \mathcal{O}_X(X) \text{ such that } f(x) = 0\}$. Show that P_x is a prime ideal in $A = \mathcal{O}_X(X)$ and the map $\theta : X \to \operatorname{Spec} A$, $x \mapsto P_x$ is continuous.

d) Prove that $\theta^{-1}\big(D(f)\big) = X_f$.
e) Show that the map θ gives rise to a morphism of schemes $X \to \operatorname{Spec} A$.

11. Let X be a quasi-projective variety such that the scheme $\widetilde{X}$ is affine. Prove that the variety X is affine.

12. Let X be a discrete topological space, k a field. For every subset $U \subset X$ let $\mathcal{F}(U)$ be the ring of all maps $U \to k$. Prove that $\mathcal{F}$ is a sheaf of rings with respect to obvious restriction maps and the ringed space $(X, \mathcal{F})$ is a scheme. For which X is this scheme affine?

13. Let K be the rational function field over a field k in infinitely many variables $X_1, X_2, \dots, X_n, \dots$. For every $i \geq 1$, let v_i be the X_i-adic discrete valuation on K, that is $v_i(f) = n$ for a nonzero function $f \in K$ if f can be written in the form $f = (X_i)^n \cdot \frac{g}{h}$ with polynomials g and h not divisible by X_i. For every nonzero function $f \in K$ we assign a sequence of integers $a(f) = (a_1, a_2, \dots)$ as follows. Let $a_1 = v_1(f)$, $f_2 = (f/X_1^{a_1})_{X_1=0}$. Set $a_2 = v_2(f_2)$, $f_3 = (f_2/X_2^{a_2})_{X_2=0}$ and $a_3 = v_3(f_3)$ and so on.
a) We order the set of sequences $a = (a_1, a_2, \dots)$ lexicographically, i.e., $a \geq b$ if for the smallest index k such that $a_k \neq b_k$ we have $a_k \geq b_k$. Prove that the set

$$A = \{0\} \cup \{f \in K \setminus \{0\} \text{ such that } a(f) \geq 0\}$$

is a local ring with the maximal ideal Q generated by all the X_i.
b)

For every $i \geq 1$, let P_i be the union of $\{0\}$ and the set of all nonzero $f \in A$ such that $a(f) > (0, \dots, 0, b_{i+1}, b_{i+2}, \dots)$ for all $b_{i+1}, b_{i+2}, \dots$. Prove that P_i is a prime ideal in A and

$$P_1 \subset P_2 \subset \cdots \subset P_n \subset \cdots \subset Q.$$

c) Prove that

$$\operatorname{Spec} A = \{0, Q, P_1, P_2, \dots, P_n, \dots\}$$

and describe the Zariski topology on $\operatorname{Spec} A$.
d) Prove that the scheme $\operatorname{Spec} A \setminus \{Q\}$ has no closed points.

14. Let $f : X \to Y$ be a morphism of schemes, $U \subset Y$ an open subset. Assume that $f(X) \subset U$. Prove that f factors as the composition of a morphism of schemes $X \to U$ and the inclusion $U \hookrightarrow Y$.

15. Let X be a scheme. Consider the presheaf $\mathcal{F}$ of rings on the topological space X defined by

$$\mathcal{F}(U) = \mathcal{O}_X(U)/Nil\big(\mathcal{O}_X(U)\big)$$

for every open $U \subset X$. Let $\mathcal{O}_X^{red}$ be the sheaf of rings associated to the presheaf $\mathcal{F}$ (see Ex. 1).
a) Prove that $X^{red} := (X, \mathcal{O}_X^{red})$ is a reduced scheme.
b) Construct a canonical morphism of schemes $i : X^{red} \to X$.
c) Prove that every morphism of schemes $Y \to X$ with Y reduced factors into the composition of a morphism $Y \to X^{red}$ with i.

16. Let X be a scheme and let $f : X \to \operatorname{Spec} \mathbb{Z}$ be the canonical morphism. Prove that for every $x \in X$, $f(x) = p\mathbb{Z}$, where p is the characteristic of the residue field $k(x)$.

17. Prove that any two quasi-projective varieties X and Y over F there is a natural isomorphism of schemes $\widetilde{Z} \simeq \widetilde{X} \times_{\operatorname{Spec} F} \widetilde{Y}$, where $Z = X \times Y$.

18. Let X be a separated scheme over an affine scheme. Let U and V be open affine subschemes of X. Prove that $U \cap V$ is also affine.

19. Let $S = \coprod_{i \geq 0} S_i$ be a graded commutative ring. Show that there is a natural morphism of schemes $\operatorname{Proj} S \to \operatorname{Spec} S_0$.

20. Let $A = F[t_0, t_1, \ldots t_n]$ be the polynomial algebra. Let $I \subset A$ be a homogeneous ideal. The *saturation of* I is defined as

$$\bar{I} = \{a \in A \mid \quad \text{for every} \quad i = 0, \ldots n \quad \text{there exists an} \quad m \quad \text{with} \quad at_i^m \in I\}.$$

Prove that $\bar{I}$ is a homogeneous ideal of A.

21. Let $A = F[t_0, t_1, \ldots t_n]$ be the polynomial algebra and let I and J be two homogeneous ideal of A. Prove that I and J define the same closed subscheme of $\operatorname{Proj} A$ if and only if they have the same saturations.