PROBLEMS, MATH 214A

Presheaves and sheaves

1. Let $p : E \to X$ be a covering. Prove that p is an open map, i.e., it takes open sets in E to the open sets in X.

2. Let A be a presheaf of sets. Prove that the canonical morphism $A \to A^{sh}$ is injective if and only if A is separated.

3. Let A be a subsheaf of a sheaf B of abelian groups. Prove that the factor presheaf B/A is separated.

4. Let A be a sheaf of abelian groups. Prove that $A = 0$ if and only if $A_x = 0$ for all $x \in X$.

5. Prove that a sequence of sheaves on X of abelian groups $0 \to A \to B \to C \to 0$ is exact if and only if the sequence of abelian groups $0 \to A_x \to B_x \to C_x \to 0$ is exact for all $x \in X$.

6. Let C be a set and X a topological space. Choose a point $x \in X$. Define a presheaf of sets Sk on X by

$$Sk(U) = \begin{cases} C, & \text{if } x \in U; \\ \{*\}, & \text{otherwise.} \end{cases}$$

a) Prove that Sk is a sheaf (called a *skyscraper sheaf*).
b) Prove that $\mathrm{Mor}_{Sh}(A, Sk) \simeq \mathrm{Mor}_{Sets}(A_x, C)$ for every sheaf of sets A.

7. Let $f : X \to Y$ be a morphism of topological spaces. Prove that $f^{-1}(B)_x \simeq B_{f(x)}$ for every sheaf B on Y.

8. Let A and B be two sheaves. Prove that the presheaf of sets $\mathit{Mor}(A, B)$ defined by

$$\mathit{Mor}(A, B)(U) := \mathrm{Mor}(A|_U, B|_U)$$

is a sheaf.

9. Let A be a presheaf of abelian groups on X and $s \in A(U)$ a section over an open subset $U \subset X$. Prove that the set

$$Supp(s) = \{x \in U \quad \text{such that} \quad s_x \neq 0\}$$

is closed in U.

10. Let $f : X \to Y$ be a morphism of topological spaces.
a) Prove that the functor $f_* : Sh(X, Ab) \to Sh(Y, Ab)$ is left exact.
b) Prove that the functor $f^* : Sh(Y, Ab) \to Sh(X, Ab)$ is exact.

Affine schemes

11. Let $\alpha : A \to B$ be a homomorphism of commutative rings, $J \subset B$ an ideal and $I = \alpha^{-1}(J)$. Prove that $V(I)$ is the closure of the set $\alpha^*(V(J))$ in $\operatorname{Spec} A$.

12. Let $\operatorname{Spec} A$ be a disjoint union of two closed subsets Z_1 and Z_2. Prove that there are ideals I_1 and I_2 in A such that $A = I_1 + I_2$, $I_1 \cap I_2 = 0$ and $Z_k = V(I_k)$ for $k = 1$ and 2.

13. Let A be a discrete valuation ring with quotient field K. Construct a morphism of ringed spaces $(\operatorname{Spec} K, \mathcal{O}_K) \to (\operatorname{Spec} A, \mathcal{O}_A)$ that is not a morphism of locally ringed spaces.

14. Describe $\operatorname{Spec} \mathbb{Z}[X]$.

15. Find an example of a commutative ring A that is not Noetherian with Noetherian topological space $\operatorname{Spec} A$.

16. Give an example of a commutative ring A and an open set $U \subset \operatorname{Spec} A$ that is not principal.

17. Prove that the functor $CommRings^{op} \to TopSpaces$, $A \mapsto \operatorname{Spec} A$ is neither full nor faithful.

Schemes

18. Let U be an open subscheme of a scheme X. Prove that the embedding $U \hookrightarrow X$ is a monomorphism in the category of schemes.

19. Let $f : X \to Y$ be a morphism of schemes, $W \subset Y$ an open subset. Assume that $f(X) \subset W$. Prove that f factors as the composition of a morphism of schemes $X \to W$ and the inclusion $W \hookrightarrow Y$.

20. Let X be a scheme and let $f : X \to \operatorname{Spec} \mathbb{Z}$ be the canonical morphism. Prove that for every $x \in X$, $f(x) = p\mathbb{Z}$, where p is the characteristic of the residue field $k(x)$.

21. Prove that the category $Schemes$ admits coproducts.

22. Let $A_1, A_2, \ldots, A_n, \ldots$ be nonzero commutative rings. Prove that the scheme $\coprod_{i=1}^{\infty} \operatorname{Spec}(A_i)$ is not affine.

23. Let X be a scheme and $x \in X$. Prove that the image of the natural morphism $\operatorname{Spec} \mathcal{O}_{X,x} \to X$ consists of all points x' such that x is contained in the closure of $\{x'\}$.

24. Let X be a scheme over K and $\mathbb{G}_m := \operatorname{Spec} K[x, x^{-1}]$. Show that the set $\operatorname{Mor}_K(X, \mathbb{G}_m)$ has a natural structure of a group. Prove that the group $\operatorname{Mor}_K(X, \mathbb{G}_m)$ is isomorphic to the group $\mathcal{O}_X(X)^{\times}$ of invertible elements in $\mathcal{O}_X(X)$.

25. Let $\mathcal{P}$ be the class of all open embeddings (respectively, closed embeddings, locally closed embeddings). Prove that $\mathcal{P}$ is local on the target.

26. Let X be the line with doubled origin over a field K. Describe the closure of the image of the diagonal morphism $X \to X \times_K X$.

27. Let X be the line with doubled origin over a field K and let R be a commutative K-algebra. Construct a bijection between the set $X(R)$ of R-points of X and the set of pairs (r, e), where $r \in R$ and e is an idempotent in R/rR.

28. Let $f : X \to Y$ be a morphism of schemes. Prove that the diagonal morphism $\Delta_X : X \to X \times_Y X$ is a locally closed embedding.

29. Prove that the class of open embeddings is closed under compositions and base changes.

30. Prove that the class of closed embeddings is closed under compositions and base changes.

31. Prove that the class of morphisms of finite type is closed under compositions and base changes.

32. Let $\alpha : S \to T$ be a graded homomorphism of graded commutative rings.
a) Prove that the set U of ideals $Q \in \mathrm{Proj}(T)$ such that $\alpha(S^+)$ is not contained in Q is open. Show that α yields a morphism $U \to \mathrm{Proj}(S)$.
b) Suppose there is an integer n such that $\alpha_i : S_i \to T_i$ is an isomorphism for all $i \geq n$. Prove that $U = \mathrm{Proj}(T)$ and the induced morphism $\mathrm{Proj}(T) \to \mathrm{Proj}(S)$ is an isomorphism.

33. Prove that open and closed embeddings are monomorphisms in $Schemes$.

34. Let F be a field, X the coproduct of countably many copies of the scheme $\mathrm{Spec}\, F$. The the ring A is the product of countably many copies of F. Prove that there is an open embedding $X \to \mathrm{Spec}\, A$.

35. Let X be a coproduct of countably many copies of the scheme $Y = \mathrm{Spec}\, \mathbb{Z}$ and let $f : X \to Y$ be (the only) morphism. Prove that the $\mathcal{O}_Y$-module $f_*(\mathcal{O}_X)$ is not quasi-coherent.

36. Let $A = F[x_1, x_2, \ldots, x_n, \ldots]$ be the polynomial ring in countably many variables over a field F, let M be the ideal generated by all variables. Prove that the scheme $\mathrm{Spec}(A) \setminus \{M\}$ is not quasi-compact.

37. Prove that the class of affine morphisms is closed under compositions, base changes and local on the target.

38. Let $f : X \to Y$ be an affine morphism and $\mathcal{A} = f_*(\mathcal{O}_X)$. Prove that f_* yields an equivalence of the category of quasi-coherent $\mathcal{O}_X$-modules and the category of $\mathcal{A}$-modules (i.e., the category of quasi-coherent $\mathcal{O}_Y$-modules that have a structure of $\mathcal{A}$-modules).

39. Prove that the class of finite morphisms is closed under compositions, base changes and local on the target.

40. Consider the following classes of morphisms:

$\mathcal{P}_1 = $ Open embeddings morphisms
$\mathcal{P}_2 = $ Closed embeddings morphisms
$\mathcal{P}_3 = $ Separated morphisms
$\mathcal{P}_4 = $ Quasi-compact morphisms
$\mathcal{P}_5 = $ Locally of finite type morphisms
$\mathcal{P}_6 = $ Finite type morphisms
$\mathcal{P}_7 = $ Proper morphisms
$\mathcal{P}_8 = $ Affine morphisms
$\mathcal{P}_9 = $ Finite morphisms

In the ij-cell of the table below write
"+" if $\mathcal{P}_i \subseteq \mathcal{P}_j$,
"−" if $\mathcal{P}_i \nsubseteq \mathcal{P}_j$ and
"?" if you are not sure.

You are expected to get at least 80% of correct answers. "?" counts toward 1/2 of a correct answer.

	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$	$\mathcal{P}_5$	$\mathcal{P}_6$	$\mathcal{P}_7$	$\mathcal{P}_8$	$\mathcal{P}_9$
$\mathcal{P}_1$	*								
$\mathcal{P}_2$		*							
$\mathcal{P}_3$			*						
$\mathcal{P}_4$				*					
$\mathcal{P}_5$					*				
$\mathcal{P}_6$						*			
$\mathcal{P}_7$							*		
$\mathcal{P}_8$								*	
$\mathcal{P}_9$									*

The end