PROBLEMS, MATH 214A

Affine and quasi-affine varieties

k is an algebraically closed field

Basic notions

1. Let $X \subset \mathbb{A}^2$ be defined by $x^2 + y^2 = 1$ and $x = 1$. Find the ideal $I(X)$.

2. Prove that the subset in $\mathbb{A}^2$ consisting of all points of the form (t^2, t^3), $t \in k$ is closed.

3. Let X and X' be subsets of $\mathbb{A}^n$. Show that $I(X) \subset I(X')$ if and only if $X' \subset \overline{X}$.

4. Let S and S' be subsets of $k[T_1, \ldots, T_n]$. Show that $Z(S) \subset Z(S')$ if and only if $S' \subset \sqrt{\langle S \rangle}$.

5. Prove that a subset $X \subset \mathbb{A}^n$ is quasi-affine if and only if $X = Z_1 \setminus Z_2$, where Z_1 and Z_2 are closed subsets of $\mathbb{A}^n$.

6. Prove that the intersection of two quasi-affine subsets is quasi-affine.

7. Let $Z = Z(x)$ and $U = \mathbb{A}^2 \setminus Z(y)$ be respectively closed and open subsets of $\mathbb{A}^2$ with the coordinates (x, y). Prove that the set $Z \cup U$ is not quasi-affine.

8. Let X be a quasi affine subset of $\mathbb{A}^n$. Prove that every closed or open subset of X is quasi-affine.

9. Prove that every quasi-affine subset of $\mathbb{A}^1$ is either open or closed.

10. Give an example of a quasi-affine subset of A^n, $n \geq 2$ that is neither open nor closed.

11. Prove that every quasi-affine set (in particular, an open set!) is quasi-compact in Zariski topology.

Regular functions and maps

12. Let $X \subset \mathbb{A}^2$ be defined by $x(y^2 - x) = 0$. Show that the function $f : X \to k$ defined by

$$f(x, y) = \begin{cases} 0 & \text{if } x = 0, \\ y & \text{otherwise} \end{cases}$$

is not regular. Prove that $f(x, y)^2$ is regular.

13. Prove that every open subset of $\mathbb{A}^1$ is principal.

14. Show that the open subset $\mathbb{A}^2 \setminus (0, 0)$ is not principal in $\mathbb{A}^2$.

15. Suppose that X consists of n points. Prove that the ring $k[X]$ is isomorphic to the direct product of n copies of k.

16. Establish a bijection between an closed set X and the set of all k-algebra homomorphisms $k[X] \to k$.

17. Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be given by $f(x, y) = (xy, y)$. Show that the image of f is not quasi-affine.

18. Let $f : X \to Y$ be a regular maps of quasi-affine sets and let $I \subset k[Y]$, $J \subset k[X]$ be two ideals. Prove that

$$f^{-1}(Z(I)) = Z(f^*(I))$$

and

$$\overline{f(Z(J))} = Z(f^{*^{-1}}(J)).$$

19. Prove that two curves X and Y given in $\mathbb{A}^2$ by equations $xy = 0$ and $x^2 = x$ respectively are not isomorphic.

20. Let x be a point in $\mathbb{A}^n$ and let $X = \mathbb{A}^n \setminus \{x\}$. Prove that if $n \geq 2$, then $k[X] = k[\mathbb{A}^n]$.

21. Let $f : X \to \mathbb{A}^1 = k$ be a regular function. Prove that the image of f is either closed or open in $\mathbb{A}^1$.

22. Let X be a quasi-affine set such that $k[X] = k$. Prove that X is a one point set.

23. Let S be a subset of $k[X]$, X a closed set. Prove that the principal open sets $D(f)$, for all $f \in S$, cover X if and only if the ideal generated by S is equal to $k[X]$.

24. Let X be a point and let $Y = \mathbb{A}^2 \setminus \{(0, 0)\}$. Show that the natural map

$$Mor(X, Y) \to Hom_{k-alg}(k[Y], k[X])$$

is not surjective. ($Mor(X, Y)$ is the set of all regular maps $X \to Y$.)

25. Let X be a quasi-affine set and Y be a closed set. Prove that the natural map

$$Mor(X, Y) \to Hom_{k-alg}(k[Y], k[X])$$

is a bijection.

26. Prove that $\mathbb{A}^1 \setminus \{0, 1\}$ and $\mathbb{A}^1 \setminus \{a, b\}$ are isomorphic for all $a \neq b \in k$.

27. Prove that $\mathbb{A}^1$ is not isomorphic to any proper quasi-affine subset of $\mathbb{A}^1$.

28. Prove that the curves $Z(x^2 + y^2 - 1)$ and $Z(xy - 1)$ in $\mathbb{A}^2$ are isomorphic (assume char $k \neq 2$). Are these curves isomorphic to $\mathbb{A}^1$?

Categories. Products and coproducts

29. Let $\mathcal{A}$ be a category and let $X, Y \in Ob(\mathcal{A})$. The *product* of X and Y in $\mathcal{A}$ is an object $X \times Y$ together with two morphisms $p : X \times Y \to X$ and $q : X \times Y \to Y$ such that for every two morphisms $f : Z \to X$ and $g : Z \to Y$ there exists a unique morphism $h : Z \to X \times Y$ with $f = ph$ and $g = qh$. Prove that the product is unique up to canonical isomorphism. Determine products in the categories of sets, rings, commutative rings, (left) modules over a ring, closed sets, quasi-affine sets.

30. Formulate the dual notion of the coproduct in a category. Determine coproducts in the categories of sets, rings, commutative rings, (left) modules over a ring, closed sets, quasi-affine sets.

31. An *initial* (resp. *final*) object of a category $\mathcal{A}$ is an object $X \in Ob(\mathcal{A})$ such that for any object $Y \in Ob(\mathcal{A})$ there is a unique morphism $X \to Y$ (resp. $Y \to X$). Prove that initial and final objects are unique up to canonical isomorphism. Determine initial and final objects in the categories of sets, rings, commutative rings, (left) modules over a ring, closed sets, quasi-affine sets.

32. Prove that for quasi-affine sets X and Y, the canonical homomorphism $k[X] \otimes k[Y] \to k[X \times Y]$ is an isomorphism.

Irreducible varieties

33. Let $X_1, X_2, \ldots, X_n$ be the irreducible components of a Noetherian topological space X.
a) Let $U \subset X$ be an open subset. Prove that the nonempty intersections $U \cap X_i$ are the irreducible components of U.
b) Prove that an open subset $U \subset X$ is dense if and only if $U \cap X_i \neq \emptyset$ for every i.
c) Prove that every irreducible subset of X is contained in the X_i for some i.
d) Prove that the irreducible components of X can be defined as maximal elements in the set of all closed irreducible subsets of X.

34. Let $f : X \to Y$ be a continuous map of topological spaces and let $Z \subset X$ be an irreducible subset. Prove that $f(Z)$ is also irreducible.

35. Find irreducible components of the variety given by $x^2 = yz$, $xz = x$ in $\mathbb{A}^3$.

36. Find irreducible components of the variety given by $xy = z^3$, $xz = y^3$ in $\mathbb{A}^3$.

37. Let X be a quasi-affine variety and let $Y \subset X$ be a closed irreducible subset. Prove that $I(Y)$ is a prime ideal in $k[X]$.

38. Show that every nonempty quasi-affine variety admits a cover by open affine dense subsets.

39. Prove that any quasi-affine variety is isomorphic to a dense open subset of an affine variety.

40. Prove that the algebra $k[X]$ for a quasi-affine variety X is isomorphic to a subalgebra of a finitely generated k-algebra.

41. Let X be an affine variety, $X = Z_1 \cup Z_2$, where Z_i are closed and disjoint. Show that there is a function $e \in k[X]$ such that $e|_{Z_1} = 0$ and $e|_{Z_2} = 1$. Prove that $k[X]$ is isomorphic to the product $k[Z_1] \times k[Z_2]$.

42. Let X be an affine variety and let $e \in k[X]$ be a function such that $e^2 = e$. Prove that $X = Z_1 \cup Z_2$, where Z_i are closed , disjoint and $e|_{Z_1} = 0$, $e|_{Z_2} = 1$.

43. Prove that an quasi-affine variety X is connected if and only if the ring $k[X]$ has no idempotents other than 0 and 1.

44. Prove that a Noetherian topological space is Hausdorf if and only if it is finite with discrete topology.

Rational functions

45. Find the domain of definition of the rational function $(1-y)/x$ on the curve given by $x^2+y^2=1$ in $\mathbb{A}^2$.

46. Prove that the variety $\mathbb{A}^1$ satisfies the following property: If $f \in k(\mathbb{A}^1)$ and $f^2 \in k[\mathbb{A}^1]$, then $f \in k[\mathbb{A}^1]$. Does the variety $Z(y^2-x^2-x^3) \subset \mathbb{A}^2$ satisfy this property?

47. Prove the the map $\alpha : \mathbb{A}^2 \to \mathbb{A}^2$ given by $\alpha(x,y) = (x, xy)$ is a birational isomorphism. Find the domain of definition of α^{-1}.

48. Prove that the field $k(x, \sqrt{1-x^2})$ is purely transcendental over k.

49. Prove that the plane curves given by the equations $y^2 = x^3$ and $y^2 = x^3 + x^2$ respectively are rational varieties.

50. Let X be "sphere" given in $\mathbb{A}^n$ by the equation $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ $(n \geq 2)$. Prove that X is a rational variety.

Local ring of a subvariety

51. Let X and X' be quasi-affine varieties and let $Y \subset X$ and $Y' \subset X'$ be closed irreducible subvarieties. Prove that the local rings $\mathcal{O}_{X,Y}$ and $\mathcal{O}_{X',Y'}$ are isomorphic as k-algebras if and only if there are neighborhoods U and U' of Y and Y' in X and X' respectively and an isomorphism $f : U \to U'$ such that $f(U \cap Y) = U' \cap Y'$.

52. Let Y and Z be closed irreducible subvarieties of X such that $Y \subset Z$. Let $P \subset \mathcal{O}_{X,Y}$ be the prime ideal corresponding to Z. Prove that $\mathcal{O}_{X,Y}/P \simeq \mathcal{O}_{Z,Y}$ and $(\mathcal{O}_{X,Y})_P \simeq \mathcal{O}_{X,Z}$.

53. The *completion* $\hat{R}$ of a local commutative ring R with maximal ideal M is the inverse limit of the rings R/M^i over all $i \geq 1$. Show that $\hat{R}$ is a local ring. Let X and Y be plane curves given by equations $xy = 0$ and $x^3 + y^3 + xy = 0$ respectively. Let $z = (0,0)$ be the origin. Prove that the local rings $\mathcal{O}_{X,z}$ and $\mathcal{O}_{Y,z}$ are not isomorphic but have isomorphic completions.

Quasi-projective varieties

Basic notions

54. Let $I \subset k[S_0, S_1, \ldots, X_n]$ be an ideal. Prove that the following are equivalent:

(i) I is generated by homogeneous polynomials;

(ii) If $F_0 + F_1 + \cdots + F_d \in I$, where F_i is a homogeneous polynomial of degree i, then $F_i \in I$ for all i.

55. Let F be a homogeneous polynomial. Prove that every divisor of F is also homogeneous.

56. Let $I \subset k[S_0, S_1, \dots, X_n]$ be a homogeneous ideal. Prove that the ideal $\sqrt{I}$ is also homogeneous.

Regular functions and maps

57. Prove that the image the Veronese map is closed.

58. Is there a surjective regular map $\mathbb{A}^1 \to \mathbb{P}^1$?

59. Let $x \in \mathbb{P}^2$. Prove that $\mathbb{P}^2 \setminus \{x\}$ is neither projective nor quasi-affine variety.

60. Determine all regular maps $\mathbb{P}^n \to \mathbb{A}^m$.

61. Let $F \in k[S_0, S_1, \dots, S_n]$ be a homogeneous polynomial of degree d and let U be the principal open set $D(F)$ in $\mathbb{P}^n$. Prove that the k-algebra $k[U]$ is canonically isomorphic to the k-algebra $k[S_0, S_1, \dots, S_n]_{(F)}$ of all rational functions $\frac{G}{F^m}$, where G is a homogeneous polynomial of degree md.

62. Prove that every regular map $f : \mathbb{P}^n \to \mathbb{P}^m$ is of the form $x \mapsto [F_0(x) : F_1(x) : \cdots : F_m(x)]$, where $F_0, F_1, \dots, F_m \in k[S_0, S_1, \dots, S_n]$ are homogeneous polynomials of the same degree having no nontrivial common zeros. Prove that every regular map $\mathbb{P}^n \to \mathbb{P}^m$ with $m < n$ is constant.

63. Prove that a regular map $f : X \to Y$ of quasi-projective varieties is an isomorphism if and only if f is a homeomorphism and for any point $x \in X$ the induced local ring homomorphism

$$f^* : \mathcal{O}_{Y,f(y)} \to \mathcal{O}_{X,x}$$

is an isomorphism.

Rational functions and maps

64. Find the domain of definition of the regular function $f = S_1/S_0$ on $\mathbb{P}^2$.

65. Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be the rational map defined by $f([S_0 : S_1 : S_2]) = [S_1S_2 : S_0S_2 : S_0S_1]$. Find the domain of definition of f. Prove that f is a birational isomorphism and $f^2 = \mathrm{id}$.

66. Prove that every rational map $\mathbb{P}^1 \to \mathbb{P}^n$ is regular.

67. Let $f : X \to Y$ be a rational map of quasi-projective varieties with X irreducible. Prove that there is a regular map $g : X' \to X$ for some quasi-projective X' such that g is a birational isomorphism and the composition $f \circ g$ is regular.

Product of quasi-projective varieties

68. Prove that $(X \times Y) \times Z \simeq X \times (Y \times Z)$.

69. Prove that $\mathbb{P}^n \times \mathbb{P}^m$ is not isomorphic to $\mathbb{P}^{n+m}$.

70. Prove that $\mathbb{A}^1 \times \mathbb{P}^1$ is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

71. Let $f : X \to S$ and $g : Y \to S$ be two regular maps. Show that the set $\{(x, y) \in X \times Y : f(x) = g(y)\}$ is a quasi-projective variety.

Proper maps

72. Prove the if $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are proper maps, then $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is proper.

73. Prove that a morphism $f : X \to Y$ is proper if and only if f factors as $X \to Y \times \mathbb{P}^n \to Y$ for some n with the first map a closed embedding and the second one the projection.

74. Let $f : X \to Y$ and $g : Y \to Z$ be two regular maps. Prove that if $g \circ f$ is proper, then f is proper.

75. Prove that a map $f : X \to Y$ is proper if and only if Y can be covered by open subsets U_i such that $f^{-1}(U_i) \to U_i$ is proper for each i.

Dimension and Smoothness

76. Let $X \subset \mathbb{A}^n$ be a closed irreducible variety of dimension $n - 1$. Prove that $X = Z(F)$ for an irreducible polynomial F.

77. Let $C \subset \mathbb{A}^{n+1}$ be a closed cone and $X \subset \mathbb{P}^n$ the corresponding variety. Prove that if X is not empty, then $\dim(C) = \dim(X) + 1$.

78. Prove that $X \times Y$ is smooth if and only if X and Y are smooth.

79. Prove that if $\mathrm{char}(k) \neq 2$, then the quadric $Z(S_0^2 + S_1^2 + \cdots + S_n^2)$ in $\mathbb{P}^n$ is smooth.