HOMEWORK 5

1. Find a nonzero abelian group A such that $A \oplus A \simeq A$. Let $R = \operatorname{End}(A)$. Prove that the (left) R-modules R and R^2 are isomorphic. Show that $R^n \simeq R^m$ for all $n, m > 0$.

2. Let R be a ring that admits a homomorphism to a division ring. Prove that the (left) R-modules R^n and R^m are isomorphic only if $n = m$.

3. Let R be a ring such that all (left) cyclic R-modules are projective. Prove that R is semisimple.

4. Let R be a ring such that every two simple (left) R-modules are isomorphic. Prove that R has no nontrivial central idempotents.

5. Determine all irreducible representations of the dihedral group of order $2n$ over $\mathbb{C}$.

6. Find the dimensions of all irreducible representations of the alternating group A_5 over $\mathbb{C}$.

7. Determine all irreducible representations of $\mathbb{Z}/n\mathbb{Z}$ over $\mathbb{Q}$.

8. Determine all irreducible representations of the symmetric group S_3 over $\mathbb{Q}$.

9. A representation $G \to \mathrm{GL}_n(F)$ over a field F is called *absolutely irreducible* if the composition $G \to \mathrm{GL}_n(F) \to \mathrm{GL}_n(\overline{F})$, where $\overline{F}$ is an algebraic closure of F, is irreducible. Prove that if $G \to \mathrm{GL}_n(F)$ is absolutely irreducible, then for every field extension L/F, the composition $G \to \mathrm{GL}_n(F) \to \mathrm{GL}_n(L)$ is irreducible.

10. Let G be a finite group, $H \subset G$ a subgroup, V an H-space. Consider the vector space W of all maps $f : G \to V$ satisfying $f(hg) = hf(g)$ for all $h \in H$ and $g \in G$. Show that the G-action on W given by $(gf)(g') = f(g'g)$ for all $g, g' \in G$ and $f \in W$ makes W an G-space.