HOMEWORK 4

1. Let M, M' and N be finitely generated modules over a Dedekind ring such that $M \oplus N \simeq M' \oplus N$. Prove that $M \simeq M'$.

2. Let M be a nonzero torsion free module over a Dedekind ring R and $I \subset R$ an ideal. Show that there is an injective R-module homomorphism $I \to M$.

3. Let I be a nonzero ideal of a Dedekind ring R. Prove that $I \oplus I^{-1} \simeq R^2$. Deduce that I is generated by two elements.

4. Let I be a nonzero ideal of a Dedekind ring R. Prove that $I^{-1} \simeq \mathrm{Hom}_R(I, R)$.

5. Let $M_1, M_2, \ldots, M_n$ be (left) R-modules and $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$. Prove that the ring $\mathrm{End}_R(M)$ is isomorphic to the ring of $n \times n$ matrices (f_{ij}) with $f_{ij} \in \mathrm{Hom}_R(M_j, M_i)$.

6. Let R be a ring. Show that every (left) R-module is free if and only if R is a division ring.

7. Let R be a ring, $S = M_n(R)$. We view R^n as a left S-module. Prove that every S-module endomorphism f of R^n is of the form $f(x) = xa$ for some fixed $a \in R$. Deduce that $\mathrm{End}_S(R^n) \simeq R^{op}$.

8. Let V be a finite dimensional left D-module over a division ring D. Prove that the ring $\mathrm{End}_D(V)$ is simple and semisimple.

9. Let V be a finite dimensional left D-module over a division ring D and $W \subset V$ a submodule. Prove that

$$\mathrm{Hom}_D(V, W) \simeq \{f \in \mathrm{End}_D(V) \quad \text{such that} \quad \mathrm{Im}(f) \subset W\}$$

is a right ideal in $\mathrm{End}_D(V)$. Prove that every right ideal in $\mathrm{End}_D(V)$ is equal to $\mathrm{Hom}_D(V, W)$ for a unique submodule $W \subset V$.

10. Let D be a division ring. Prove that the center of D is a subfield of D.