HOMEWORK 3

1. Prove that a Dedekind ring with finitely many maximal ideals is a PID.

2. Let $R = F[x, y]/(y^2 - x^3)$, where F is a field. Determine the integral closure of R in its quotient field.

3. Let R be a normal domain with $2 \in R^\times$, F the quotient field of R and $L = F(\sqrt{d})$ a quadratic field extension of F, where $d \in R$ is nonzero.
a) Prove that an element $u + v\sqrt{d}$ with $u, v \in F$ is integral over R if and only if $u \in R$ and $dv^2 \in R$.
b) Suppose that R is a Dedekind ring and the ideal dR is not divisible by the square of a prime ideal. Prove that the ring $R[\sqrt{d}]$ is Dedekind.

4. Let F be a field of characteristic not 2, $a, b \in F^\times$. Prove that the factor ring $F[x, y]/(ax^2 + by^2 - 1)$ is Dedekind.

5. Let M be a module over a Dedekind ring R such that $aM = 0$ for some nonzero a in R. Let S be the set of all elements in R that do not belong to every prime ideal that contains a. Prove that S is a multiplicative subset of R and the localization $S^{-1}R$ is a PID. Show that M has a natural structure of a $S^{-1}R$-module. Deduce the structure theorems on finitely generated torsion modules over R.

6. Let R be a Dedekind ring and $a \in R$ a nonzero element. Prove that the ring R/aR is Artinian.

7. Let I and J be nonzero ideals of a Dedekind ring R. Prove that the R-modules I/IJ and R/J are isomorphic.

8. Let R be a ring of algebraic integers, $p \in \mathbb{Z}$ a prime integer. Prove that the set A of all prime ideals $P \subset R$ such that $P \cap \mathbb{Z} = p\mathbb{Z}$ is finite and nonempty. Show that $pR = \prod P^{e_P}$ for some positive integers e_P, where the product is taken over all $P \in A$.

9. In the conditions of the previous problem let n be the degree of the quotient field of R over $\mathbb{Q}$. Show that for every $P \in A$, the factor ring R/P is a finite field of p^{f_P} elements for some integer f_P. Prove that $n = \sum e_P f_P$, where the sum is taken over all $P \in A$. (Hint: Compute the number of elements in R/pR.)

10. Let $\alpha = (1+\sqrt{-23})/2$ and I the ideal in $\mathbb{Z}[\alpha]$ generated by 2 and α. Prove that I^3 is a principal ideal.