HOMEWORK 2

1. Prove that every ideal in a Dedekind ring can be generated by two elements.

2. Is $\mathbb{Z}[\sqrt{5}]$ a Dedekind ring?

3. Prove that a Noetherian integral domain R is a Dedekind ring if and only if each localization R_P at a prime ideal P is a DVR (discrete valuation ring).

4. Show that $R := \mathbb{Q}[X,Y]/(X^2+Y^2-1)\mathbb{Q}[X,Y]$ is a Dedekind ring. Is R a PID? (Hint: Compare R with the subring $\mathbb{Q}[X]$.)
5. Prove that for every two ideals A and B of a Dedekind ring one has $AB = (A+B)(A\cap B)$.

6. Prove that the ring of rational functions $f/g \in F(x)$ with $\deg(f) \leq \deg(g)$ is a DVR.

7. Let $d \neq 0, 1$ be an integer. Suppose that d is not divisible by the square of a prime integer. Let $R = \mathbb{Z}[\frac{1+\sqrt{d}}{2}]$ if $d \equiv 1$ modulo 4 and $R = \mathbb{Z}[\sqrt{d}]$ otherwise. Prove that R is a Dedekind ring.

8. Let p be prime integer. Prove that if $p \equiv 3$ modulo 4, then pR is a prime ideal in the ring of Gauss integers $R = \mathbb{Z}[i]$. Prove that if $p \equiv 1$ modulo 4, then pR is a product of two distinct prime ideals of R. Find a prime ideal P in R such that $2R = P^2$.

9. Let $R = \mathbb{Z}[\sqrt{-5}]$. Factor $14R$ into a product of prime ideals.

10. Let R be a Dedekind ring and $S \subset R$ a multiplicative subset. Prove that the localization $S^{-1}R$ is also a Dedekind ring.