HOMEWORK 1

1. Prove that an Artinian integral domain is a field.

2. Let $f : M \to M$ be a surjective endomorphism of a Noetherian module M over a commutative ring. Show that f is an isomorphism.

3. Let M be a Noetherian module over a commutative ring R. Show that for every multiplicative subset $S \subset R$, the $S^{-1}R$-module $S^{-1}M$ is Noetherian.

4. Let F be an algebraically closed field and let $M \subset F[X_1, \ldots, X_n]$ be a maximal ideal. Prove that there exist $a_1, \ldots, a_n \in F$ such that
$M = \{f \in F[X_1, \ldots, X_n] : f(a_1, \ldots, a_n) = 0\}$.

5. Let F be a field and let $M \subset F[X_1, \ldots, X_n]$ be a maximal ideal. Prove that factor ring $F[X_1, \ldots, X_n]/M$ is a finite field extension of F.

6. Show that the ring of all 2×2 matrices $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ such that $a \in \mathbb{Z}$ and $b, c \in \mathbb{Q}$ is right Noetherian but not left Noetherian.

7. Show that the ring of all 2×2 matrices $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ such that $a \in \mathbb{Q}$ and $b, c \in \mathbb{R}$ is right Artinian but not left Artinian.

8. Let I be an ideal of a commutative ring R. Set

$$\sqrt{I} = \{a \in R \text{ such that } a^n \in I \text{ for some } n \in \mathbb{N}\}.$$

Prove that $\sqrt{I}$ is an ideal in R (called the *radical of* I).

9. Let F be an algebraically closed field, $I \subset F[X_1, X_2, \ldots, X_n]$ an ideal. Denote by $S(I)$ the subset in F^n consisting of all n-tuples $(a_1, a_2, \ldots, a_n) \in F^n$ such that $f(a_1, a_2, \ldots, a_n) = 0$ for all $f \in I$. A subset $S \subset F^n$ is called *closed* if $S = S(I)$ for some ideal $I \subset F[X_1, X_2, \ldots, X_n]$. Prove that the union of two closed subsets and the intersection of any family of closed subsets is also closed.

10. For any closed subset $S \subset F^n$ denote by $I(S)$ the set of all polynomials $f \in F[X_1, X_2, \ldots, X_n]$ such that $f(a_1, a_2, \ldots, a_n) = 0$ for all $(a_1, a_2, \ldots, a_n) \in S$. Show that $I(S)$ is an ideal in $F[X_1, X_2, \ldots, X_n]$ such that $I(S) = \sqrt{I(S)}$ (i.e. $I(S)$ is a *radical* ideal).