Math 210C Name:
Take-Home Final
due Friday, June 8, 2018

1. a) Let F be a field and $R = F[[t]]$ the ring of formal power series $\sum_{i\geq 0} a_i t^i$ with $a_i \in F$. Prove that R is a DVR.

b) Find a discrete valuation of the field of rational functions $\mathbb{Q}(x, y)$ with the residue field isomorphic to $\mathbb{Q}$.

2. Prove that the ring $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$ is a PID.

3. Let A be an algebra over a field F of zero characteristic and G a finite subgroup of the multiplicative group $A^\times$. Prove that if G spans A as a vector space over F, then A is a semisimple ring.

4. Let V be a finite dimensional vector space over an algebraically closed field F and $A \subset \mathrm{End}_F(V)$ an F-subalgebra. Prove that if V, viewed as a left A-module via the inclusion $A \hookrightarrow \mathrm{End}_F(V)$, is a simple A-module, then $A = \mathrm{End}_F(V)$.

5. Let $\chi_1, \chi_2, \ldots, \chi_k$ be all irreducible characters of a finite group G over $\mathbb{C}$ and $g_1, g_2, \ldots, g_k$ representatives of all conjugacy classes $C_1, C_2, \ldots, C_k$, respectively. Let A be the $k \times k$ matrix $(\chi_i(g_j))$. Prove that

$$\det(A)^2 = \frac{(-1)^{\frac{s}{2}}|G|^k}{|C_1||C_2|\cdots|C_k|},$$

where s is the number of indices $i = 1, 2, \ldots, k$ such that $C_i^{-1} \neq C_i$.

6. Let G be the symmetric group S_3. Prove that $\mathbb{Q}[G] \simeq \mathbb{Q} \times \mathbb{Q} \times M_2(\mathbb{Q})$.

7. Determine all irreducible representations over $\mathbb{R}$ of the quaternion group Q_8.

8. Let χ be the character of a representation of a finite group G over $\mathbb{C}$. Prove that the function $G \to \mathbb{C}$ taking g to $\chi(g^2)$ is a linear combination of irreducible characters with integer coefficients.

9. Let A be a central simple F-algebra of dimension n^2. Prove that $A \simeq M_n(F)$ if and only if A has a left ideal I with $\dim_F(I) = n$.

10. Let A and B be central division algebras over a field F. Prove that if $\dim(A)$ and $\dim(B)$ are relatively prime, then $A \otimes_F B$ is a division algebra.