HOMEWORK 8

1. Let $A = (a, b)$ be the quaternion algebra over a field F of characteristic different from 2. Prove that $A \simeq M_2(F)$ is and only if the equation $ax^2 + by^2 = z^2$ has a nonzero solution in F.

2. Let $A_1 = (a, b_1)$ and $A_2 = (a, b_2)$ be two quaternion algebras over a field F of characteristic different from 2. Prove that the algebra $A_3 = (a, b_1 b_2)$ is isomorphic to a subalgebra of $A_1 \otimes A_2$.

3. In the conditions of problem 2 compute the centralizer of A_3 in $A_1 \otimes A_2$. Deduce that $[A_3] = [A_1] + [A_2]$ in $Br(F)$.

4. Prove that the endomorphism σ of the quaternion algebra $A = (a, b)$ over a field F of characteristic different from 2 given by $\sigma(x + yi + zj + tk) = x - yi - zj - tk$ is an involution of A, i.e., $\sigma(uv) = \sigma(v)\sigma(u)$ for all $u, v \in A$ and $\sigma \circ \sigma = \mathrm{id}_A$.

5. In the setup of problem 4 prove that every element $u \in A$ is a root of the quadratic polynomial $t^2 - (u + \sigma(u))t + u\sigma(u)$ over F.

6. Let A be an F-algebra. Prove that A is isomorphic to the algebra $M_n(B)$ for some n and an F-algebra B if and only if there are elements $e_{ij} \in A$ for $i, j = 1, \dots n$ satisfying $\sum_i e_{ii} = 1$ and $e_{ij} e_{kl} = \delta_{jk} e_{il}$.

7. Let L/F be a field extension of degree n. Prove that L is isomorphic to an F-subalgebra of $M_n(F)$.

8. Let A and B be two F-algebras such that B is simple and $M_n(A)$ is isomorphic to a subalgebra of $M_n(B)$. Prove that A is isomorphic to a subalgebra of B. (Hint: consider the centralizer of $M_n(F)$ in $M_n(A)$ and in $M_n(B)$.)

9. Let A be a central simple F-algebra of degree n and L/F a field extension of degree kn such that L is isomorphic to a subalgebra of $M_k(A)$. Prove that L is a splitting field of A.

10. Let A be a central division F-algebra of degree n and L/F a splitting field of degree s. Prove that n divides s and L is isomorphic to a subalgebra of $M_{s/n}(A)$.