HOMEWORK 7

The base field F is algebraically closed of characteristic zero and G is a finite group.

1. Let χ be an irreducible character of a group G of order n. Prove that $\sum_{g\in G}\chi(g)^2$ is equal to 0 or n.

2. Let V and W be two G-spaces. Define the structure of a G-space on $\mathrm{Hom}(V,W)$ by $(gf)(v) = gf(g^{-1}v)$ for $g \in G$, $f \in \mathrm{Hom}(V,W)$ and $v \in V$. Prove that

$$\dim \mathrm{Hom}_G(V,W) = \langle \chi_\rho, \chi_\mu \rangle,$$

where $\rho : G \to \mathrm{GL}(V)$ and $\mu : G \to \mathrm{GL}(W)$ are the representations.

3. Let V be a simple G-space. Show that there is at most one (up to a scalar multiple) G-invariant non-degenerate bilinear form on V.

4. For an arbitrary G-space V and a 1-dimensional G-space L show that $W \otimes L$ is simple if and only if V is simple.

5. Find all finite groups that have exactly two non-isomorphic irreducible representations.

6. Let $\rho : G \to \mathrm{GL}(V)$ a representation. Show that ρ yields a representation $\mu : G \to \mathrm{GL}(S^2V)$, where $S^2V := (V \otimes T)/span\{v \otimes w - w \otimes v\}$ is the symmetric square of V. Prove that

$$\chi_\mu(g) = \frac{1}{2}\big(\chi_\rho(g)^2 + \chi_\rho(g^2)\big).$$

7. Let $\rho : S_n \to GL_n$ be the natural representation of the symmetric group S_n by permutations of the coordinates. Determine $\langle \chi_\rho, \chi_\rho \rangle$.

8. A representation $\rho : G \to \mathrm{GL}(V)$ is called faithful if ρ is injective. Find a faithful representation of the quaternion group Q_8 of the smallest dimension.

9. Let G be a p-group and H the subgroup of all central elements g with $g^p = e$. Let ρ be a representation of G. Prove that if the restriction $\rho|_H$ of ρ on H is faithful, then ρ is faithful.

10. Let $\rho : G \to \mathrm{GL}(V)$ be a faithful representation of G. Prove that every simple G-space W is a direct summand of the tensor product of several copies of V.