HOMEWORK 6

1. Let G be a finite group and C the center of G. Let $\mu : C \to F^\times$ be a character of C. Prove that there is an irreducible representation $\rho : G \to \mathrm{GL}(V)$ such that $\rho(c)(v) = \mu(c)v$ for all $c \in C$ and $v \in V$.

2. Let F be a field of characteristic $p > 0$ and G a finite p-group. Prove that $\mathrm{rad}(F[G]) = \{\sum a_g g \in F[G] \mid \sum a_g = 0\}$. Determine all simple (left) $F[G]$-modules.

3. Let V be the kernel of $F^n \to F$ taking $(a_1, \dots, a_n)$ to $\sum a_i$. The symmetric group S_n acts on V by permutations of the coordinates. Prove that if the characteristic of F does not divide n, then the corresponding representation is irreducible.

4. Let ρ be a representation of a finite group G and V the corresponding G-space. Show that the dual space V^* has the structure of a G-space via $(g\varphi)(v) = \varphi(g^{-1}v)$ for $g \in G$, $\varphi \in V^*$ and $v \in V$. Prove that $\chi_{\rho^*}(g) = \chi(g^{-1})$, where ρ^* is the representation corresponding to the G-space V^*.

5. Prove that the F-algebras $F[G \times H]$ and $F[G] \otimes_F F[H]$ are isomorphic.

For all problems below the base field F is algebraically closed of characteristic zero.

6. Let G be a finite group. Define the abelian group $Rep(G)$ by generators and relations as follows. The generators are the isomorphism classes $[\rho]$ of representations ρ of G. The relations are $[\rho \oplus \rho'] = [\rho] + [\rho']$ for all representations ρ and ρ'. Prove that $Rep(G)$ is a free abelian group with basis the set of isomorphism classes of irreducible representations of G. Prove that the tensor product yields the structure of a commutative ring on $Rep(G)$.

7. Let G be a finite group. Prove that the map $Rep(G) \to Ch(G)$ taking the class $[\rho]$ to the character χ_ρ is a well defined injective ring homomorphism. Write the multiplication table for $Rep(S_3)$.

8. Prove that every irreducible representation of the product of two finite groups G and H is isomorphic to $\rho \otimes \mu$, where ρ and μ are irreducible representations of G and H respectively.

9. Find an irreducible 2-dimensional representation of the quaternion group Q_8.

10. Let $C(g)$ be the conjugacy class of an element g in a finite group G of order n and χ be the character of an irreducible representation ρ. Prove that if $|C(g)|$ is relatively prime to n and $\chi(g) \neq 0$, then $\rho(g)$ is the multiplication by a scalar.