Math 210C
Take-Home Final
due Wednesday, June 1, 2016

Name:

1. Let A be an algebra over a field F of zero characteristic and G a finite subgroup of the multiplicative group $A^{\times}$. Prove that if G spans A as a vector space over F, then A is a semisimple ring.

2. Let K/F be a Galois field extension of degree n. Prove that the ring $K \otimes_F K$ is isomorphic to the product of n copies of K.

3. Let K/F be a Galois field extension with $Gal(K/F) = G$, $H \subset G$ a subgroup of G and $E = K^H$. Prove that $Gal(E/F) = N_G(H)/H$, where $N_G(H)$ is the normalizer of H in G.

4. Let V be a finite dimensional vector space over an algebraically closed field F and $A \subset \mathrm{End}_F(V)$ an F-subalgebra. Prove that if V, viewed as a left A-module via the inclusion $A \to \mathrm{End}_F(V)$, is a simple A-module, then $A = \mathrm{End}_F(V)$.

5. Let A be a central simple F-algebra of dimension n^2. Prove that $A \simeq M_n(F)$ if and only if A has a left ideal I with $\dim_F(I) = n$.

6. Let V be a simple finite dimensional G-space for a finite group G. Construct an isomorphism of G-spaces $V \otimes V$ and $\mathrm{Hom}(V^*, V)$. Prove that the dual G-space V^* is isomorphic to V if and only if there is a nonzero G-invariant vector in the G-space $V \otimes V$.

7. Let A and B be central division algebras over a field F. Prove that if $\dim(A)$ and $\dim(B)$ are relatively prime, then $A \otimes_F B$ is a division algebra.

8. Find a discrete valuation on the field of rational functions $\mathbb{Q}(x, y)$ with the residue field isomorphic to $\mathbb{Q}$.

9. Let G be the symmetric group S_3. Prove that $\mathbb{Q}[G] \simeq \mathbb{Q} \times \mathbb{Q} \times M_2(\mathbb{Q})$.

10. Determine all irreducible representations over $\mathbb{R}$ of the quaternion group of order 8.