HOMEWORK 9

1. Let V and W be inner product spaces of finite dimension such that $\dim(V) \leq \dim(W)$. Prove that there is a linear map $T : V \to W$ such that $\langle T(v), T(v') \rangle = \langle v, v' \rangle$ for all $v, v' \in V$.

2. Let $T : V \to V$ be an orthogonal projection on a subspace of an inner product space V. Prove that $||T(v)|| \leq ||v||$ for all $v \in V$.

3. Let $T : V \to V$ be a projection on a subspace of an inner product space V such that $||T(v)|| \leq ||v||$ for all $v \in V$. Prove that T is an orthogonal projection.

4. Let T be a normal operator on a finite dimensional inner product space. Prove that if $T : V \to V$ is a projection, then T is an orthogonal projection.

5. Let T be a normal operator on a finite dimensional complex inner product space such that $T^n = 0$ for some $n > 0$. Prove that $T = 0$.

6. Let T be a normal operator on a finite dimensional complex inner product space. Prove that for any integer $n > 0$, there exists a linear operator S such that $T = S^n$.

7. Let T be a unitary operator on a finite dimensional inner product space and $W \subset V$ a finite dimensional T-invariant subspace. Prove that $W^\perp$ is also T-invariant.

8. Find an orthogonal matrix whose first row is $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$.

9. Let T be a unitary operator on a complex finite dimensional inner product space V. Prove that there exists a unitary operator S on V such that $T = S^2$.

10. Let T be a self-adjoint positive definite operator on a finite dimensional inner product space V. Prove that there is an operator S on V such that $T = S^* \cdot S$.