HOMEWORK 8

1. Let $a, b \in F$ be nonzero elements such that $a + b \neq 0$. Let $\{v, w\}$ be an orthogonal basis of a 2-dimensional vector space with a quadratic form Q such that $Q(v) = a$ and $Q(w) = b$. Prove that there is another orthogonal basis $\{v', w'\}$ such that $Q(v') = a + b$ and $Q(w') = ab(a + b)$.

2. Determine all isometries of the quadratic form $Q(x, y) = xy$ on F^2.

3. Let B be a symmetric bilinear form on a vector space V and $W \subset V$ a subspace. Prove that $((W^\perp)^\perp)^\perp = W^\perp$.

4. Prove that $B(X, Y) = trace(XY)$ is a bilinear form on $M_{2\times 2}(F)$. Is B non-degenerate? (The trace of a square matrix is the sum of all diagonal elements.)

5. Let S be a subset of an inner product vector space V and $W = span(S)$. Prove that $v \in W^\perp$ if and only if $v \perp w$ for all $w \in S$.

6. Let T be a linear operator on an inner product vector space V such that $||T(v)|| = ||v||$ for all $v \in V$. Prove that T is injective.

7. Let T be a linear operator on an inner product vector space V such that $||T(v)|| = ||v||$ for all $v \in V$. Prove that $\langle T(v), T(u)\rangle = \langle v, u\rangle$ for all $v, u \in V$.

8. A self-adjoint linear operator T on a finite dimensional inner product vector space V is called *positive definite* if $\langle T(v), v\rangle > 0$ for every nonzero $v \in V$. Prove that an operator T is positive definite if and only if all eigenvalues of T are positive.

9. Let T be a linear operator on a finite dimensional inner product vector space V. Prove that if T is an isomorphism, then the operators TT^* and T^*T are positive definite.

10. Let T be a self-adjoint positive definite operator on an inner product vector space V. Prove that the formula $\langle v, u\rangle' = \langle T(v), u\rangle$ defines another inner product on V.