HOMEWORK 7

1. Let B be a bilinear form on a finite dimensional vector space V, β a basis for V. Prove that B is symmetric if and only if the matrix $[B]_\beta$ is symmetric.

2. Let B be a bilinear form on a finite dimensional vector space V. Suppose that for any nonzero $v \in V$ there exists a $u \in V$ such that $B(v, u) \neq 0$. Prove that for any nonzero $u \in V$ there exists a $v \in V$ such that $B(v, u) \neq 0$.

3. Let B be a bilinear form on a finite dimensional vector space V, β a basis for V. Let $T_B : V \to V^*$ be the corresponding linear map. Prove that $([B]_\beta)^t = [T_B]_\beta^{\beta^*}$, where β^* is the dual basis for V^*.

4. Let B be a symmetric bilinear form on a finite dimensional vector space V. Prove that there is a unique bilinear form $\overline{B}$ on the factor space $V/V^\perp$ such that $\overline{B}(\bar{v}, \bar{u}) = B(v, u)$ for all $v, u \in V$.

5. Let B be a bilinear form on a finite dimensional vector space V, β and γ two bases for V. Prove that $[B]_\gamma = C^t \cdot [B]_\beta \cdot C$, where C is the change of basis matrix.

6. Show that all quadratic forms on a vector space V of dimension n form a vector space $Quad(V)$. Determine $\dim Quad(V)$.

7. Let Q be quadratic form on a real vector space V of dimension n such that $Q(v) > 0$ for all nonzero $v \in V$. Prove that there is an orthogonal basis $\{v_1, v_2, \ldots, v_n\}$ for V such that $Q(v_i) = 1$ for all i.

8. Let Q be the quadratic form on $\mathbb{R}^3$ defined by

$$Q(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_3x_1.$$

Find an orthogonal basis for $\mathbb{R}^3$.

9. Let Q be the quadratic form on $P_n(\mathbb{R})$ defined by $Q(f) = f(0)^2$. Find an orthogonal basis for $P_n(\mathbb{R})$.

10. Let Q be the quadratic form on $M_{2\times 2}(\mathbb{R})$ defined by $Q(A) = \det(A)$. Find an orthogonal basis for $M_{2\times 2}(\mathbb{R})$.