HOMEWORK 4

1. Find all monic divisors of the polynomial $X(X^2-1)(X+1)$ over $\mathbb{Q}$.

2. Find polynomials f and g over $\mathbb{R}$ such that $(X-1)^2 \cdot f + (X+1)^2 \cdot g = 1$.

3. Find the remainder on dividing X^3 by X^2+1.

4. Let $A \in M_{n \times n}(\mathbb{C})$ and $f \in \mathbb{C}[X]$ a polynomial relatively prime to the minimal polynomial m_A. Prove that the matrix $f(A)$ is invertible.

5. Find a matrix A over $\mathbb{R}$ that is not diagonalizable, but A is diagonalizable over $\mathbb{C}$.

6. Let T be an operator in a vector space V of finite dimension over F such that $(T - \lambda \cdot Id_V)^k = 0$ for some $\lambda \in F$ and $k > 0$. Prove that T is diagonalizable if and only if $T = \lambda \cdot Id_V$.

7. Let W and W' be subspaces of a vector space V such that $V = W \oplus W'$. Prove that the vector spaces V/W and W' are isomorphic.

8. Let V be a vector space, $W \subset V$ a subspace and $v_1, v_2, \ldots, v_n \in V$. Prove that if $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \in V/W$ are linearly independent, then $v_1, v_2, \ldots, v_n$ are linearly independent in V.

9. Let V be a vector space and $W \subset V$ a subspace. Prove that the map $T : V \to V/W$ defined by $T(v) = \bar{v}$ is linear. Determine $N(T)$ and $R(T)$.

10. Let $T : V \to V$ be a linear operator and $W \subset V$ a T-invariant subspace. Prove that there is a linear map $\overline{T} : V/W \to V/W$ such that $\overline{T}(\bar{v}) = \overline{T(v)}$ for all $v \in V$.