HOMEWORK 2

1. Let V be a vector space and let $\Phi_V : V \to V^{**}$ be the canonical linear map. Let $T : V \to W$ be a linear map. Prove that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\Phi_V} & V^{**} \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T^{**}} \\ W & \xrightarrow{\Phi_W} & W^{**} \end{array}$$

is commutative, i.e., the compositions $\Phi_W \circ T$ and $T^{**} \circ \Phi_V$ coincide.

2. Let $W_1, W_2, \ldots, W_n$ be subspaces of a vector space V. For every $i = 1, 2, \ldots, n$, let Z_i be the subspace $W_1 + \cdots + W_{i-1} + W_{i+1} + \cdots + W_n$ of V. Prove that the sum $W_1 + W_2 + \cdots + W_n$ is direct if and only if $W_i \cap Z_i = \{0\}$ for every i.

3. Let $V_1, V_2, \ldots, V_n$ be vector spaces over a field F. Prove that the component-wise addition and scalar multiplication makes the product $V = V_1 \times V_2 \times \cdots \times V_n$ a vector space over F. For every $i = 1, 2, \ldots, n$, let W_i be the subspace of $V = V_1 \times V_2 \times \cdots \times V_n$ consisting of all tuples $(v_1, v_2, \ldots, v_n)$ such that $v_j = 0$ for all $j \neq i$. Prove that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$.

4. Let $W_1, W_2, \ldots, W_n$ be subspaces of a vector space V such that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$. Let $T_i : W_i \to Z$ be linear maps, $i = 1, 2, \ldots, n$. Prove that there is a unique linear map $T : V \to Z$ such that $T(w_1 + w_2 + \cdots + w_n) = T_1(w_1) + T_2(w_2) + \cdots + T_n(w_n)$ for all $w_i \in W_i$.

5. Let V be a finite dimensional vector space and $W \subset V$ a subspace. Prove that there is a subspace $W' \subset V$ such that $V = W \oplus W'$.

6. Let $V = W_1 + W_2$ and $W = W_1 \cap W_2$. Prove that
$V/W = (W_1/W) \oplus (W_2/W)$.

7. Let W and W' be subspaces of a vector space V such that $V = W \oplus W'$. Prove that the vector spaces V/W and W' are isomorphic.

8. Let V be a vector space, $W \subset V$ a subspace and $v_1, v_2, \ldots, v_n \in V$. Prove that if $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \in V/W$ are linearly independent, then $v_1, v_2, \ldots, v_n$ are linearly independent in V.

9. Let V be a vector space and $W \subset V$ a subspace. Prove that the map $T : V \to V/W$ defined by $T(v) = \bar{v}$ is linear. Determine $N(T)$ and $R(T)$.

10. Let $T : V \to V'$ be a linear map. Prove that the spaces $V/N(T)$ and $R(T)$ are isomorphic.