HOMEWORK 6

1. Let V be a real vector space of real-valued functions spanned by e^t, te^t, t^2e^t and e^{2t}. Find a Jordan canonical basis and a Jordan canonical form of the operator T on V defined by $T(f) = f'$.

2. Find a Jordan canonical basis and a Jordan canonical form of the operator T on $P_3(\mathbb{R})$ defined by $T(f(x)) = xf''(x)$.

3. Find a Jordan canonical basis and a Jordan canonical form of the operator T on $P_3(\mathbb{R})$ defined by $T(f) = f'' + 2f$.

4. Find a Jordan canonical basis and a Jordan canonical form of the operator T on $M_{2\times 2}(\mathbb{R})$ defined by

$$T(A) = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \cdot A - A^t.$$

5. Find a Jordan canonical basis and a Jordan canonical form of the operator T on $M_{2\times 2}(\mathbb{R})$ defined by

$$T(A) = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \cdot (A - A^t).$$

6. Let V be a real vector space of polynomials in two variables x and y of degree at most 2. Find a Jordan canonical basis and a Jordan canonical form of the operator T on V defined by

$$T(f(x, y)) = \frac{\partial}{\partial x} f(x, y) + \frac{\partial}{\partial y} f(x, y).$$

7. Let $A \in M_{n\times n}(F)$ be such that the characteristic polynomial P_A is split. Prove that the matrices A and A^t are similar.

8. Let T be a linear operator in a finite dimensional vector space V such that the characteristic polynomial of T is split, and let $\lambda_1, \lambda_2, \dots, \lambda_k$ be all eigenvalues of T. Let $S : V \to V$ be the linear map defined by

$$S(v) = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_k v_k,$$

where for each i, v_i is the unique generalized eigenvector for the eigenvalue λ_i, such that $v = v_1 + v_2 + \ldots + v_k$. Prove that S is a diagonalizable operator and the operator $T - S$ is nilpotent.

9. Let P_1 and P_2 be two projectors in vector space V such that $P_1 P_2 = P_2 P_1$. Prove that $P_1 P_2$ is also projector and $R(P_1 P_2) = R(P_1) \cap R(P_2)$.

10. Let $P : V \to V$ be a projector in a vector space V of dimension n. Prove that there is a basis β for V such that

$$[P]_\beta = \begin{pmatrix} I_k & 0 \\ 0 & 0_{n-k} \end{pmatrix}$$