HOMEWORK 5

1. Let T be a linear operator in a vector space of finite dimension. Prove that T is nilpotent (i.e., T^m is the zero operator for some $m > 0$) if and only if the matrix of T is some basis is strictly upper triangular (all elements on the main diagonal and below are zero).

2. Let T be a nilpotent operator in a vector space of dimension n. Prove that T is nilpotent if and only if $P_T = X^n$.

3. Let T be a nilpotent operator in a vector space of dimension n. Prove that $T^n = 0$.

4. Let T be a nilpotent operator in a vector space of finite dimension. Suppose that $R(T^m) = R(T^{m+1})$ for some $m > 0$. Prove that $T^m = 0$.

5. Find the Jordan canonical form and the Jordan canonical basis for the linear operator on $P_2(\mathbb{R})$ defined by $T(f) = 2f - f'$.

6. Find the Jordan canonical form and the Jordan canonical basis for the linear operator on $M_{2\times 2}(F)$ defined by $T(A) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot A$.

7. Find the Jordan canonical form and the Jordan basis for the linear operator on $M_{2\times 2}(F)$ defined by $T(A) = 2A + A^t$.

8. Find the Jordan canonical form of the matrix $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -1 & 3 \end{pmatrix}$

9. Let J be the Jordan canonical form of a linear operator T. Find the Jordan canonical form of the operator aT, where a is a nonzero scalar.

10. Let W be a subspace of a vector space V. Let $T : V \to V/W$ be the linear operator defined by $T(v) = \bar{v}$. Consider the dual operator $T^* : (V/W)^* \to V^*$. Prove that $N(T^*) = \{0\}$ and $R(T^*)$ coincides with the null space of the restriction operator $V^* \to W^*$.