HOMEWORK 3

1. Let T be a linear operator on a vector space V over F and let f be a polynomial over F. Prove that the subspaces $N(f(T))$ and $R(f(T))$ of V are T-invariant.

2. Let $A \in M_{n\times n}(F)$. Show that the Cayley-Hamilton Theorem implies that $\dim span(I_n, A, A^2, A^3, \ldots) \leq n$.

3. Let $A \in M_{n\times n}(F)$. Prove that $\dim span(I_n, A, A^2, A^3, \ldots) = \deg(m_A)$, where m_A is the minimal polynomial of A.

4. Let $A \in M_{n\times n}(F)$ and let $T = L_A : F^n \to F^n$ be the linear operator of left multiplication by A, i.e., $T(X) = AX$. Prove that $m_T = m_A$.

5. Let $T : F^2 \to F^2$ be the linear operator defined by $T(a, b) = (a+b, a-b)$. Determine the minimal polynomial m_T.

6. Determine m_A for the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

7. Let T be the linear operator on the space $M_{n\times n}(F)$ defined by $T(A) = A^t$. Determine m_T.

8. Let T be a linear operator on a vector space V of dimension n. Suppose that the characteristic polynomial P_T splits. Prove that P_T divides $(m_T)^n$.

9. Let T be a linear operator on a finite dimensional vector space V and let $W \subset V$ be a T-invariant subspace. Let $S : W \to W$ be the restriction of T on W. Prove that m_S divides m_T.

10. Let T be an invertible linear operator on a finite dimensional vector space V over F. Prove that there exists a polynomial $f \in F[X]$ such that $T^{-1} = f(T)$.