HOMEWORK 2

1. Let V be a vector space and let $\Phi_V : V \to V^{**}$ be the canonical linear map. Let $T : V \to W$ be a linear map. Prove that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\Phi_V} & V^{**} \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T^{**}} \\ W & \xrightarrow{\Phi_W} & W^{**} \end{array}$$

is commutative.

2. Let $W_1, W_2, \ldots, W_n$ be subspaces of a vector space V. For every $i = 1, 2, \ldots, n$, let Z_i be the subspace $W_1 + \cdots + W_{i-1} + W_{i+1} + \cdots + W_n$ of V. Prove that the sum $W_1 + W_2 + \cdots + W_n$ id direct if and only if $W_i \cap Z_i = \{0\}$ for every i.

3. Let $V_1, V_2, \ldots, V_n$ be vector spaces over a field F. Prove that the component-wise addition and scalar multiplication makes the product $V = V_1 \times V_2 \times \cdots \times V_n$ a vector space over F. For every $i = 1, 2, \ldots, n$, let W_i be the subspace of $V = V_1 \times V_2 \times \cdots \times V_n$ consisting of all tuples $(v_1, v_2, \ldots, v_n)$ such that $v_j = 0$ for all $j \neq i$. Prove that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$.

4. Let $W_1, W_2, \ldots, W_n$ be subspaces of a vector space V such that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$. Let $T_i : W_i \to Z$ be linear maps, $i = 1, 2, \ldots, n$. Prove that there is a unique linear map $T : V \to Z$ such that $T(w_1 + w_2 + \cdots + w_n) = T_1(w_1) + T_2(w_2) + \cdots + T_n(w_n)$ for all $w_i \in W_i$.

5. Let $T : V \to V$ be a linear operator in a vector space V. Prove that the subspaces $N(T)$ and $R(T)$ of V are T-invariant.

6. Let T be the differentiation operator in the space $P_n(\mathbb{R})$ of all polynomials over $\mathbb{R}$ of degree at most n, i.e., $T(f) = f'$ for every f. Determine all T-invariant subspaces in $P_n(\mathbb{R})$.

7. Let $T : V \to V$ be a linear operator in a vector space V of dimension n. Prove that T is diagonalizable if and only if there are T-invariant 1-dimensional subspaces $W_1, W_2, \ldots, W_n \subset V$ such that
$V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$.

8. Let $T : V \to V$ be a linear operator in a vector space V of dimension n. Prove that there is a flag of T-invariant subspaces

$$\{0\} = W_0 \subset W_1 \subset \cdots \subset W_n = V$$

with $\dim(W_i) = i$ for all i if and only if there exists a basis β for V such that the matrix $[T]_\beta$ is upper triangular (i.e., all the elements below the diagonal are zero).

9. Let $T : V \to V$ be a linear operator in a vector space V. Prove that for every vector $v \in V$ there is a T-invariant subspace $W_v \subset V$ such that W_v is contained in every T-invariant subspace of V that contains v.

10. Let $T : V \to V$ be a linear operator in a vector space V and $W \subset V$ a T-invariant subspace. Let $v_1, v_2, \ldots, v_n$ be eigenvectors of T with distinct eigenvalues. Prove that if $v_1 + v_2 + \cdots + v_n \in W$, then $v_i \in W$ for all i.