HOMEWORK 1

1. Let V be a vector space, S a set and $s \in S$. Consider the subset W in the vector space U of all maps $f : S \to V$ such that $f(s) = 0$. Is W a subspace of U?

2. Prove that $span\{(1, -1, 0), (0, 1, -1)\}$ coincides with the subspace of $\mathbb{R}^3$ consisting of all vectors (a, b, c) such that $a + b + c = 0$.

3. Let S be a linearly dependent subset of a vector space V. Let S' be the subset of S consisting of all vectors in S that are linear combinations of other vectors in S. For any $n > 0$, find the smallest value of $card(S')$ over all vector spaces V and all subsets $S \subset V$ with $card(S) = n$.

4. Find the dimension of $span\{X^2 - 1, (X - 1)^2, X - 1\}$ in $P_2[X]$.

5. Let $T : V \to W$ be a linear map. Show that if the vectors $v_1, v_2, \ldots, v_n$ span V, then the vectors $T(v_1), T(v_2), \ldots, T(v_n)$ span the range $R(T)$.

6. Let $T : V \to V$ be a linear map such that $N(T^2) \neq 0$. Show that $N(T) \neq 0$.

7. Let V be a vector space of dimension n and $W \subset V$ a subspace of dimension $n - 1$. Prove that there is an $f \in V^*$ such that $W = N(f)$.

8. Let $\{f_1, f_2, \ldots, f_n\}$ be the dual basis of a basis $\{v_1, v_2, \ldots, v_n\}$. Find the dual basis of the basis $\{v_1 + v_2, v_2, \ldots, v_n\}$.

9. Let $T : V \to W$ be a linear map. Prove that $\dim N(T) + \dim R(T^*) = \dim(V)$.

10. Let $S = \{v_1, v_2, \ldots, v_m\}$ be a basis of a subspace W of a vector space V. Let $S' = \{v_1, v_2, \ldots, v_m, \ldots, v_n\}$ be a basis of V and $\{f_1, f_2, \ldots, f_n\}$ the dual basis of S'. Write g_i for the restriction of the linear function f_i on W. Prove that $\{g_1, g_2, \ldots, g_m\}$ is the dual basis of S.