HOMEWORK 8

1. Let V be an inner product space. Show that $||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2$ for all $x, y \in V$.

2. Prove that $B(X,Y) = \det(X+Y) - \det(X) - \det(Y)$ is a bilinear form on $M_{2\times 2}(\mathbb{R})$. Is B an inner product?

3. Let V be an inner product space and $x, y \in V$. Prove that $|\langle x, y\rangle| = ||x|| \cdot ||y||$ if and only if one of the vectors x or y is a multiple of the other.

4. Let V be an inner product space and $v, w \in V$. Prove that $|(||x|| - ||y||)| \leq ||x-y||$.

5. Apply the Gram-Schmidt process to the subset $\{(1,0,1),(0,1,1),(1,3,3)\}$ of $\mathbb{R}^3$.

6. Let $S = \{(1,0,i),(1,2,1)\}$ in $\mathbb{C}^3$. Find a basis for $S^\perp$.

7. Let V be an inner product space and W a finite dimensional subspace of V. Prove that $(W^\perp)^\perp = W$.

8. Let W_1 and W_2 be subspaces of a finite dimensional inner product space V. Prove that $(W_1+W_2)^\perp = (W_1)^\perp \cap (W_2)^\perp$ and $(W_1 \cap W_2)^\perp = (W_1)^\perp + (W_2)^\perp$.

9. Let V be an inner product space and W a finite dimensional subspace of V. Prove that $V = W \oplus W^\perp$.

10(*). Let V be a real inner product space and $S = \{v_1, v_2, \ldots, v_n\} \subset V$. Prove that S is linearly independent if and only if the determinant of the matrix

$$\begin{pmatrix} \langle v_1, v_1\rangle & \langle v_1, v_2\rangle & \langle v_1, v_3\rangle & \ldots & \langle v_1, v_{n-1}\rangle & \langle v_1, v_n\rangle \\ \langle v_2, v_1\rangle & \langle v_2, v_2\rangle & \langle v_2, v_3\rangle & \ldots & \langle v_2, v_{n-1}\rangle & \langle v_2, v_n\rangle \\ \langle v_3, v_1\rangle & \langle v_3, v_2\rangle & \langle v_3, v_3\rangle & \ldots & \langle v_3, v_{n-1}\rangle & \langle v_3, v_n\rangle \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \langle v_{n-1}, v_1\rangle & \langle v_{n-1}, v_2\rangle & \langle v_{n-1}, v_3\rangle & \ldots & \langle v_{n-1}, v_{n-1}\rangle & \langle v_{n-1}, v_n\rangle \\ \langle v_n, v_1\rangle & \langle v_n, v_2\rangle & \langle v_n, v_3\rangle & \ldots & \langle v_n, v_{n-1}\rangle & \langle v_n, v_n\rangle \end{pmatrix}$$

is nonzero.