HOMEWORK 7

1. Prove that for any $A \in M_{n\times n}(F)$, the matrices A and A^t have the same eigenvalues.

2. Let λ be an eigenvalue of a linear operator $\mathcal{A}$. Prove that for any $m \geq 1$, λ^m is an eigenvalue of $\mathcal{A}^m$.

3. Let $\mathcal{A}$ be a diagonalizable linear operator on a vector space V. Prove that the operator $a_n\mathcal{A}^n + a_{n-1}\mathcal{A}^{n-1} + \ldots + a_1\mathcal{A} + a_0 I_V$ on V is also diagonalizable for any scalars $a_0, a_1, \ldots, a_n$.

4. Determine all diagonalizable 2×2 matrices over the a field F consisting of two elements 0 and 1.

5. Prove that if a matrix $A \in M_{n\times n}(F)$ has n distinct eigenvalues, then A is diagonalizable.

6. Give an example of a matrix $A \in M_{n\times n}(\mathbb{R})$ that is not diagonalizable, but A is diagonalizable viewed as a matrix over the field of complex numbers $\mathbb{C}$.

7. Let W_1 be a subspace of a finite dimensional vector space V. Prove that there is subspace $W_2 \subset V$ such that $V = W_1 \oplus W_2$.

8. Let W_1 and W_2 be subspaces of a vector space V such that $V = W_1 \oplus W_2$. Prove that for every subspace V' of V containing W_1 one has
$V' = W_1 \oplus (V' \cap W_2)$.

9. Let $W_1, W_2, \ldots, W_k$ be subspaces of a finite dimensional vector space V such that $V = W_1 + W_2 + \ldots + W_k$. Prove that $V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$ if and only if $\dim(V) = \sum \dim(W_i)$.

10(*). Let $\mathcal{A}$ be a linear operator such that the operator $\mathcal{A}^2$ is diagonalizable. Is $\mathcal{A}$ necessarily diagonalizable?