HOMEWORK 6

1. Prove that the system of linear equations $AX = B$ has a solution if and only if $B \in R(L_A)$.

2. Let $A \in M_{n\times n}(F)$. Suppose that the system of linear equations $AX = B$ has more than one solution. Prove that there is a column $C \in F^n$ such that the system of linear equations $AX = C$ is inconsistent.

3. Let $A \in M_{m\times n}(\mathbb{Q})$ and $B \in \mathbb{Q}^m$, where $\mathbb{Q}$ is the field of rational numbers. Suppose that the system of linear equations $AX = B$ has a solution in $\mathbb{R}^n$, where $\mathbb{R}$ is the field of real numbers. Does it necessarily have a solution in $\mathbb{Q}^n$?

4. Let B be a bilinear form on a finite dimensional vector space V. Suppose that for any nonzero vector $x \in V$ there exists a $y \in V$ such that $B(x, y) \neq 0$. Prove that for any linear function $f \in V^*$ there exists an $x \in V$ such that $f(y) = B(x, y)$ for all $y \in V$.

5. Give an example of a nonzero alternating bilinear form on the space $P_1(F)$ over F.

6. Prove that every n-linear alternating form on a vector space of dimension less than n is the zero form.

7. Prove that $\det(aA) = a^n \det(A)$ for any $A \in M_{n\times n}(F)$.

8. Let $A \in M_{n\times n}(F)$ such that $rank(A) < n$. Prove that $\det(A) = 0$.

9. Let $A \in M_{n\times n}(\mathbb{R})$ be a skew-symmetric matrix, i.e., $A^t = -A$. Prove that if n is odd, then $\det(A) = 0$.

10(*). Evaluate $\det(A)$, where A is the $n \times n$ matrix defined by $a_{ij} = \min\{i, j\}$.