HOMEWORK 5

1. Let $\mathcal{A} : V \to W$ and $\mathcal{B} : W \to Z$ be linear maps. Prove that $(\mathcal{B} \circ \mathcal{A})^* = \mathcal{A}^* \circ \mathcal{B}^*$.

2. Let $\mathcal{A} : V \to W$ be a linear map of finite dimensional vector spaces. Let $\varphi_1 : V \to V^{**}$ and $\varphi_2 : W \to W^{**}$ be canonical isomorphisms. Prove that $\mathcal{A}^{**} \circ \varphi_1 = \varphi_2 \circ \mathcal{A}$, where $\mathcal{A}^{**} = (\mathcal{A}^*)^*$ is the linear map from V^{**} to W^{**}.

3. For a subspace W of a vector space V define

$$W^0 = \{f \in V^* \mid f(v) = 0 \quad \text{for all} \quad v \in W\}.$$

Show that W^0 is a subspace of V^* and prove that $\dim(W) + \dim(W^0) = \dim(V)$.

4. For subspaces W_1 and W_2 of a finite dimensional vector space V show that $(W_1 + W_2)^0 = (W_1)^0 \cap (W_2)^0$ and $(W_1 \cap W_2)^0 = (W_1)^0 + (W_2)^0$.

5. Let $\mathcal{A} : V \to W$ be a linear map of finite dimensional vector spaces. Prove that $\mathcal{A}$ is surjective (respectively, injective) if and only if $\mathcal{A}^*$ is injective (respectively, surjective).

6. Prove that a matrix $A \in M_{n\times n}(F)$ is invertible if and only if $rank(A) = n$.

7. Let $A, B \in M_{m\times n}(F)$. Prove that $rank(A + B) \le rank(A) + rank(B)$.

8. Let $A \in M_{m\times n}(F)$ be a matrix with $rank(A) = m$. Prove that there exists a matrix $B \in M_{n\times m}(F)$ such that $AB = I_m$.

9. Let $A \in M_{m\times n}(F)$ and $B \in M_{n\times m}(F)$ be two matrices such that $AB = I_m$. Prove that $n \ge m$.

10(*). Let $A \in M_{m\times n}(F)$. Prove that $rank(A) \le 1$ if and only if there exist $X \in M_{m\times 1}(F)$ and $Y \in M_{1\times n}(F)$ such that $A = XY$.