HOMEWORK 4

1. Let $L_A : \mathbb{R}^3 \to \mathbb{R}^3$ be left multiplication by

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Find $\dim N(L_A)$ and $\dim R(L_A)$.

2. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be linear maps $V \to V$. Prove that $(\mathcal{A} \circ \mathcal{B}) \circ \mathcal{C} = \mathcal{A} \circ (\mathcal{B} \circ \mathcal{C})$ and $\mathcal{A} \circ (\mathcal{B} + \mathcal{C}) = \mathcal{A} \circ \mathcal{B} + \mathcal{A} \circ \mathcal{C}$.

3. Let $\mathcal{A} : V \to V$ be a linear map. Prove that $\mathcal{A}^2 = 0$ if and only if $R(\mathcal{A}) \subset \mathcal{N}(\mathcal{A})$.

4. Let $\mathcal{A} : V \to W$ be a linear map of vector spaces and let $\{x_1, x_2, \ldots, x_n\}$ be a basis for V. Prove that $\mathcal{A}$ is an isomorphism if and only if $\{\mathcal{A}(x_1), \mathcal{A}(x_2), \ldots, \mathcal{A}(x_n)\}$ is a basis for W.

5. Prove that an $n \times n$ matrix A over a field F is invertible if and only if the columns of A form a basis for F^n.

6. Let $\mathcal{A} : V \to W$ and $\mathcal{B} : W \to Z$ be linear maps. Prove that $N(\mathcal{A}) \subset N(\mathcal{B} \circ \mathcal{A})$ and $R(\mathcal{B} \circ \mathcal{A}) \subset R(\mathcal{B})$.

7. Let $\mathcal{A} : V \to V$ be a linear map and $\dim(V) < \infty$. Prove that there is $n > 0$ such that $N(\mathcal{A}^n) = N(\mathcal{A}^{n+1})$.

8. Let $\mathcal{A} : V \to V$ be a linear map and $\dim(V) < \infty$. Prove that if $rank(\mathcal{A}^2) = rank(\mathcal{A})$, then $N(\mathcal{A}) \cap R(\mathcal{A}) = \{0\}$.

9. Prove that if A and B are the change of coordinate matrices that change S-coordinates to T-coordinates and T-coordinates to R-coordinates, respectively, then BA is the change of coordinate matrix that changes S-coordinates to R-coordinates.

10(*). Let V be a vector space of dimension n. Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ be linear maps $V \to V$ such that $rank(\mathcal{A}_i^2) = rank(\mathcal{A}_i) = 1$ for all $i = 1, \ldots, m$ and $\mathcal{A}_i \circ \mathcal{A}_j$ is the zero map for all $i \neq j$. Prove that $m \leq n$. (Hint: Choose nonzero vectors $v_i \in R(\mathcal{A}_i)$ for all $i = 1, \ldots, m$ and show that $\mathcal{A}_i(v_i) = a_i v_i$ for some nonzero a_i and $\mathcal{A}_i(v_j) = 0$ for all $i \neq j$. Prove that the set $\{v_1, \ldots, v_m\}$ is linearly independent.)