HOMEWORK 3

1. Let $\mathcal{A} : P_n(\mathbb{R}) \to P_n(\mathbb{R})$ be the map defined by $\mathcal{A}(f) = 2f + x \cdot f'$. Is $\mathcal{A}$ linear?

2. Give an example of a linear map $\mathcal{A} : P_4(\mathbb{R}) \to P_4(\mathbb{R})$ such that $N(\mathcal{A}) = P_1(\mathbb{R})$ and $R(\mathcal{A}) = P_2(\mathbb{R})$.

3. Let $\{v_1, v_2, \ldots, v_n\}$ be a linearly independent set in a finite dimensional vector space V and let $w_1, w_2, \ldots, w_n$ be arbitrary vectors in a vector space W. Prove that there is a linear map $\mathcal{A} : V \to W$ such that $\mathcal{A}(v_i) = w_i$ for all $i = 1, 2, \ldots, n$.

4. Let V and W be finite dimensional vector spaces over F and $V' \subset V$, $W' \subset W$ two subspaces. Prove that if $\dim(V') + \dim(W') = \dim(V)$, then there is a linear map $\mathcal{A} : V \to W$ such that $N(\mathcal{A}) = V'$ and $R(\mathcal{A}) = W'$.

5. Find an isomorphism between $M_{3\times 2}(F)$ and $P_5(F)$.

6. Let V be a finite dimensional vector space over F with basis S and let $v \in V$. Find a linear map $\mathcal{A} : F \to V$ such that $[\mathcal{A}]_T^S = [v]_S$, where $T = \{1\}$ is the basis for F.

7. Let $\mathcal{A} : M_{2\times 2}(F) \to M_{2\times 2}(F)$ be the map defined by $\mathcal{A}(A) = A + A^t$, where A^t is the transpose of A. Compute $[\mathcal{A}]_S^S$, where S is the basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

8. Let V and W be finite dimensional vector spaces over F and $W' \subset W$ a subspace. Show that the subset of $\mathcal{L}(V, W)$ consisting of all linear maps $\mathcal{A} : V \to W$ such that $R(\mathcal{A}) \subset W'$ is a subspace of $\mathcal{L}(V, W)$.

9. Let V and W be finite dimensional vector spaces with $\dim(V) = \dim(W)$ and let $\mathcal{A} : V \to W$ be a linear map. Prove that there are bases S of V and T of W such that the matrix $[\mathcal{A}]_S^T$ is diagonal.

10. (*) Let V be a finite dimensional vector space and let $\mathcal{A} : V \to V$ be a linear map such that $\mathcal{A} \circ \mathcal{A} = \mathcal{A}$. Prove that there is a basis S of V such that

$$[\mathcal{A}]_S^S = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$$

for some k, where I_k and 0 are the identity and zero matrices respectively.