HOMEWORK 1

1. In every vector space V over a field F, show that $(a-b)(x-y) = ax - ay - bx + by$ for all $x, y \in V$ and $a, b \in F$.

2. Let $V = F^2$ for F a field. For $(a_1, a_2), (b_1, b_2) \in V$ and $c \in F$, define $(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2)$ and $c(a_1, a_2) = (ca_1, ca_2)$. Is V a vector space over F with respect to these operations?

3. Let $C^1(\mathbb{R})$ be the set of all real-valued functions defined on the real line that have continuous derivative. Prove that $C^1(\mathbb{R})$ is a subspace of the space of real-valued functions defined on the real line.

4. Let W_1 and W_2 be two subspaces of a vector space V. Show that $W_1 + W_2$ is the intersection of all subspaces of V that contain W_1 and W_2.

5. A matrix $M \in \mathsf{M}_{n\times n}(\mathbb{R})$ is called *symmetric* (respectively, *skew-symmetric*) if $M^t = M$ (respectively, $M^t = -M$).

a) Prove that the sets $\mathsf{Sym}_{n\times n}(\mathbb{R})$ and $\mathsf{Skew}_{n\times n}(\mathbb{R})$ of all symmetric and skew-symmetric matrices in $\mathsf{M}_{n\times n}(\mathbb{R})$ respectively are subspaces of $\mathsf{M}_{n\times n}(\mathbb{R})$.

b) Show that $\mathsf{Sym}_{n\times n}(\mathbb{R}) + \mathsf{Skew}_{n\times n}(\mathbb{R}) = \mathsf{M}_{n\times n}(\mathbb{R})$.

6. Show that a subset S of a vector space V is a subspace of V if and only if $\mathrm{span(S)} = \mathrm{S}$.

7. Prove or disprove the following: For any two subsets S and S' of a vector space V one has $\mathrm{span(S)} \cap \mathrm{span(S')} = \mathrm{span(S \cap S')}$ and $\mathrm{span(S)} + \mathrm{span(S')} = \mathrm{span(S \cup S')}$.

8. Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other.

9. Prove that a nonempty subset S of a vector space V is linearly dependent if and only if there exist distinct vectors $v, u_1, u_2, \ldots, u_n$ in S such that v is a linear combination of $u_1, u_2, \ldots, u_n$.

10. (*) Let S be a nonempty linearly independent subset of a vector space V. Prove that for any nonzero vector $v \in V$, there is a vector $x \in S$ such that the set $(S \setminus \{x\}) \cup \{v\}$ is linearly independent.