

HOMEWORK 8

1. Let V be an inner product space. Show that $\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$ for all $v, w \in V$.
2. Prove that $B(X, Y) = \det(X + Y) - \det(X) - \det(Y)$ is a bilinear form on $M_{2 \times 2}(\mathbb{R})$. Is B an inner product?
3. Let V be an inner product space and $v, w \in V$. Prove that $|\langle v, w \rangle| = \|v\| \cdot \|w\|$ if and only if one of the vectors v or w is a multiple of the other.
4. Let V be an inner product space and $v, w \in V$. Prove that $|\|v\| - \|w\|| \leq \|v - w\|$.
5. Apply the Gram-Schmidt process to the subset $\{(1, 0, 1), (0, 1, 1), (1, 3, 3)\}$ of \mathbb{R}^3 .
6. Let $S = \{(1, 0, i), (1, 2, 1)\}$ in \mathbb{C}^3 . Find a basis for S^\perp .
7. Let V be an inner product space and W a finite dimensional subspace of V . Prove that $(W^\perp)^\perp = W$.
8. Let W_1 and W_2 be subspaces of a finite dimensional inner product space V . Prove that $(W_1 + W_2)^\perp = (W_1)^\perp \cap (W_2)^\perp$ and $(W_1 \cap W_2)^\perp = (W_1)^\perp + (W_2)^\perp$.
9. Let V be an inner product space and W a finite dimensional subspace of V . Prove that $V = W \oplus W^\perp$.
- 10(*). Let V be a real inner product space and $S = \{v_1, v_2, \dots, v_n\} \subset V$. Prove that S is linearly independent if and only if the determinant of the matrix

$$\begin{pmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \langle v_1, v_3 \rangle & \dots & \langle v_1, v_{n-1} \rangle & \langle v_1, v_n \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \langle v_2, v_3 \rangle & \dots & \langle v_2, v_{n-1} \rangle & \langle v_2, v_n \rangle \\ \langle v_3, v_1 \rangle & \langle v_3, v_2 \rangle & \langle v_3, v_3 \rangle & \dots & \langle v_3, v_{n-1} \rangle & \langle v_3, v_n \rangle \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \langle v_{n-1}, v_1 \rangle & \langle v_{n-1}, v_2 \rangle & \langle v_{n-1}, v_3 \rangle & \dots & \langle v_{n-1}, v_{n-1} \rangle & \langle v_{n-1}, v_n \rangle \\ \langle v_n, v_1 \rangle & \langle v_n, v_2 \rangle & \langle v_n, v_3 \rangle & \dots & \langle v_n, v_{n-1} \rangle & \langle v_n, v_n \rangle \end{pmatrix}$$

is nonzero.