

HOMEWORK 7

1. Prove that for any $A \in M_{n \times n}(F)$, the matrices A and A^t have the same eigenvalues.
2. Let λ be an eigenvalue of a linear operator T . Prove that for any $m \geq 1$, λ^m is an eigenvalue of T^m .
3. Let T be a diagonalizable operator on a vector space V . Prove that the operator $a_n T^n + a_{n-1} T^{n-1} + \dots + a_1 T + a_0 Id_V$ on V is also diagonalizable for any scalars a_0, a_1, \dots, a_n .
4. List all diagonalizable 2×2 matrices over the a field F consisting of two elements 0 and 1.
5. Prove that if a matrix $A \in M_{n \times n}(F)$ has n distinct eigenvalues, then A is diagonalizable.
6. Give an example of a matrix $A \in M_{n \times n}(\mathbb{R})$ that is not diagonalizable, but A is diagonalizable viewed as a matrix over the field of complex numbers \mathbb{C} .
7. Let W_1 be a subspace of a finite dimensional vector space V . Prove that there is subspace $W_2 \subset V$ such that $V = W_1 \oplus W_2$.
8. Let W_1 and W_2 be subspaces of a vector space V such that $V = W_1 \oplus W_2$. Prove that for every subspace V' of V containing W_1 one has $V' = W_1 \oplus (V' \cap W_2)$.
9. Let W_1, W_2, \dots, W_k be subspaces of a finite dimensional vector space V such that $V = W_1 + W_2 + \dots + W_k$. Prove that $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$ if and only if $\dim(V) = \sum \dim(W_i)$.
- 10(*). Let T be a linear operator such that the operator T^2 is diagonalizable. Is T necessarily diagonalizable?