

HOMEWORK 4

1. Let

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

be the matrix of a linear map T . Find $\dim N(T)$ and $\dim R(T)$.

2. Let S, T and U be linear maps $V \rightarrow V$. Prove that $(S \circ T) \circ U = S \circ (T \circ U)$ and $S \circ (T + U) = S \circ T + S \circ U$.

3. Let $T : V \rightarrow V$ be a linear map. Prove that $T^2 = T_0$ (the zero linear map) if and only if $R(T) \subset N(T)$.

4. Let $T : V \rightarrow W$ be a linear map of vector spaces and let $\{v_1, v_2, \dots, v_n\}$ be a basis for V . Prove that T is an isomorphism if and only if $\{T(v_1), T(v_2), \dots, T(v_n)\}$ is a basis for W .

5. Prove that an $n \times n$ matrix A over a field F is invertible if and only if the columns of A form a basis for F^n .

6. Let $T : V \rightarrow W$ and $S : W \rightarrow Z$ be linear maps. Prove that $N(T) \subset N(S \circ T)$ and $R(S \circ T) \subset R(S)$.

7. Let $T : V \rightarrow V$ be a linear map and $\dim(V) < \infty$. Prove that there is $n > 0$ such that $N(T^n) = N(T^{n+1})$.

8. Let $T : V \rightarrow V$ be a linear map and $\dim(V) < \infty$. Prove that if $\text{rank}(T^2) = \text{rank}(T)$, then $N(T) \cap R(T) = \{0\}$.

9. Prove that if A and B are the change of coordinate matrices that change α -coordinates to β -coordinates and β -coordinates to γ -coordinates, respectively, then BA is the change of coordinate matrix that changes α -coordinates to γ -coordinates.

10(*). Let V be a vector space of dimension n . Let T_1, T_2, \dots, T_m be linear maps $V \rightarrow V$ such that $\text{rank}(T_i^2) = \text{rank}(T_i) = 1$ for all $i = 1, \dots, m$ and $T_i \circ T_j$ is the zero map for all $i \neq j$. Prove that $m \leq n$. (Hint: Choose nonzero vectors $v_i \in R(T_i)$ for all $i = 1, \dots, m$ and show that $T_i(v_i) = a_i v_i$ for some nonzero a_i and $T_i(v_j) = 0$ for all $i \neq j$. Prove that the set $\{v_1, \dots, v_m\}$ is linearly independent.)