HOMEWORK 3

1. Let $T : P_n(\mathbb{R}) \to P_n(\mathbb{R})$ be the map defined by $T(f(x)) = 2f(x) + xf'(x)$. Is T linear?

2. Give an example of a linear map $T : P_4(\mathbb{R}) \to P_4(\mathbb{R})$ such that $N(T) = P_1(\mathbb{R})$ and $R(T) = P_2(\mathbb{R})$.

3. Let $\{v_1, v_2, \ldots, v_n\}$ be a linearly independent set in a finite dimensional vector space V and let $w_1, w_2, \ldots, w_n$ be arbitrary vectors in a vector space W. Prove that there is a linear map $T : V \to W$ such that $T(v_i) = w_i$ for all $i = 1, 2, \ldots, n$.

4. Let V and W be finite dimensional vector spaces over F and $V' \subset V$, $W' \subset W$ two subspaces. Prove that if $\dim(V') + \dim(W') = \dim(V)$, then there is a linear map $T : V \to W$ such that $N(T) = V'$ and $R(T) = W'$.

5. Find an isomorphism between $M_{3\times 2}(F)$ and $P_5(F)$.

6. Let V be a finite dimensional vector space over F with basis γ and let $v \in V$. Find a linear map $T : F \to V$ such that $[T]_\beta^\gamma = [v]_\gamma$, where $\beta = \{1\}$ is the basis for F.

7. Let $T : M_{2\times 2}(F) \to M_{2\times 2}(F)$ be the map defined by $T(A) = A + A^t$. Compute $[T]_\beta^\beta$, where β is the basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

8. Let V and W be finite dimensional vector spaces over F and $W' \subset W$ a subspace. Show that the subset of $\mathcal{L}(V, W)$ consisting of all linear maps $T : V \to W$ such that $R(T) \subset W'$ is a subspace of $\mathcal{L}(V, W)$.

9. Let V and W be finite dimensional vector spaces with $\dim(V) = \dim(W)$ and let $T : V \to W$ be a linear map. Prove that there are bases β of V and γ of W such that the matrix $[T]_\beta^\gamma$ is diagonal.

10. (*) Let V be a finite dimensional vector space and let $T : V \to V$ be a linear map such that $T \circ T = T$. Prove that there is a basis β of V such that

$$[T]_\beta^\beta = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$$

for some k, where I_k and 0 are the identity and zero matrices respectively.