

HOMEWORK 6 (due November 6)

1. Prove that the system of linear equations  $AX = B$  has a solution if and only if  $B \in R(L_A)$ .
2. Let  $A \in M_{n \times n}(F)$ . Suppose that the system of linear equations  $AX = B$  has more than one solution. Prove that there is a column  $C \in M_{n \times 1}(F)$  such that the system of linear equations  $AX = C$  is inconsistent.
3. Let  $A \in M_{m \times n}(\mathbb{Q})$  and  $B \in M_{m \times 1}(\mathbb{Q})$ . Suppose that the system of linear equations  $AX = B$  has a solution in  $\mathbb{R}^n$ . Does it necessarily have a solution in  $\mathbb{Q}^n$ ?
4. Let  $B$  be a bilinear form on a finite dimensional vector space  $V$ . Suppose that for any nonzero vector  $v \in V$  there exists a  $w \in V$  such that  $B(v, w) \neq 0$ . Prove that for any linear functional  $f \in V^*$  there exist a  $v \in V$  such that  $f(w) = B(v, w)$  for all  $w \in V$ .
5. Give an example of a nonzero alternating bilinear form on the space  $P_1(F)$  over  $F$ .
6. Prove that every  $n$ -linear alternating form on a vector space of dimension less than  $n$  is the zero form.
7. Prove that  $\det(aA) = a^n \det(A)$  for any  $A \in M_{n \times n}(F)$ .
8. Let  $A \in M_{n \times n}(F)$  such that  $\text{rank}(A) < n$ . Prove that  $\det(A) = 0$ .
9. Let  $A \in M_{n \times n}(\mathbb{R})$  be a skew-symmetric matrix. Prove that if  $n$  is odd, then  $\det(A) = 0$ .
- 10(\*). Evaluate  $\det(A)$ , where  $A$  is the  $n \times n$  matrix defined by  $a_{ij} = \min\{i, j\}$ .