HOMEWORK 8

1. Determine the Galois group of the polynomial $X^4 + 1$ over $\mathbb{Q}$.

2. Determine the Galois group of the polynomial $X^4 + 4$ over $\mathbb{Q}$.

3. Let p be a prime integer, F a field, $a \in F^\times$. Suppose that the polynomial $X^p - 1$ is split over F. Prove that the polynomial $X^p - a$ is either split over F or irreducible.

4. Let $F(\alpha)/F$ be a field extension of prime degree p. Suppose that the minimal polynomial f of α over F has at least two roots in $F(\alpha)$. Prove that f is split over $F(\alpha)$.

5. Determine the Galois group of $\mathbb{Q}(\sqrt{6}, \sqrt{10}, \sqrt{14}, \sqrt{15}, \sqrt{21}, \sqrt{35})/\mathbb{Q}$.

6. Let K/F be a field extension of prime degree p and L/F a normal closure of K/F. Prove that $[L : K]$ is not divisible by p.

7. Let f and g be nonconstant polynomials over a field F. Prove that $Gal(fg)$ is isomorphic to a subgroup of $Gal(f) \times Gal(g)$.

8. Let f be a polynomial solvable by radicals. Show that the polynomial $g(X) = f(X^2)$ is also solvable by radicals.

9. Prove that there is a cyclic field extension of $\mathbb{Q}$ of degree 1000.

10. Determine all prime integers p such that the congruence class $[3]$ is a square in $\mathbb{Z}/p\mathbb{Z}$.