HOMEWORK 7

1. Let $E$ be a subfield in $F(X)$ properly containing $F$. Show that $F(X)$ is algebraic over $E$.

2. Prove that any automorphism of $\mathbb{R}$ is the identity. (Hint: Show that for any $\sigma \in Gal(\mathbb{R}/\mathbb{Q})$, $\sigma(a) \geq a$, $a \in \mathbb{R}$.)

3. Let $K/F$ be a Galois extension, $G = Gal(K/F)$. Prove that for every $\alpha, \beta \in K$ that have the same minimal polynomial over $F$ there is $\sigma \in G$ such that $\sigma(\alpha) = \beta$.

4. Show that the fields $Q(\sqrt{2}, \sqrt{3})$ and $Q(\sqrt{2}, \sqrt{5})$ are not isomorphic.

5. Let $K = \mathbb{Q}(\alpha)$ with $\alpha$ a root of $X^3 + X^2 - 2X - 1 \in \mathbb{Q}[X]$. Show that $K/\mathbb{Q}$ is normal.

6. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \alpha)$ where $\alpha^2 = (9 - 5\sqrt{3})(2 - \sqrt{2})$. Show that $K/\mathbb{Q}$ is normal and determine $Gal(K/\mathbb{Q})$.

7. Suppose that $K/F$ is Galois. Let $F \subset E \subset K$ and $L$ the smallest subfield of $K$ containing $E$ and such that $L/F$ is normal. Prove that

$$Gal(K/L) = \bigcap_{\sigma \in Gal(K/F)} \sigma Gal(K/E) \sigma^{-1}.$$

8. Let $K/F$ be an extension of finite fields.
a) Prove that the norm map $N : K \to F$ is surjective.
b) Let $F$ be a finite field. Prove that any element in $F$ is a sum of two squares.

9. Let $X^3 + pX + q$ be an irreducible polynomial over a finite field $F$ of characteristic 3. Show that $-p$ is a square in $F$.

10. Let $K$ be a subfield of $\mathbb{R}$, $f$ an irreducible polynomial over $K$ of degree 4. Suppose that $f$ has exactly two real roots. Show that the Galois group of $f$ is either $S_4$ or of order 8.