HOMEWORK 5

1. Determine the Galois group of the extension $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})/\mathbb{Q}$ and all subfields of K.

2. Determine the Galois group of the extension $\mathbb{Q}(\sqrt[8]{2}, i)/\mathbb{Q}(\sqrt{-2})$.

3. Determine the splitting field of the polynomial $X^4 - 2X^2 - 2$ over $\mathbb{Q}$ and its Galois group.

4. Determine the splitting field of the polynomial $X^6 + 3$ over $\mathbb{Q}$ and its Galois group.

5. Let F be a field. Prove that $F(X, Y)/F(XY, X + Y)$ is a Galois extension and determine its Galois group.

6. Let F be a field of characteristic different from 2. Prove that $F(X, Y)/F(X^2, XY, Y^2)$ is a Galois extension and determine its Galois group.

7. Let K/F be a Galois extension of degree p^n (p is prime, $n > 0$). Show that there is a subfield $L \subset K$ such that L/F is a cyclic extension of degree p.

8. Let K/F be a Galois extension with $Gal(K/F) = G$. For any $\alpha \in K$ define the *norm* of α to be

$$N(\alpha) = \prod_{\sigma \in G} \sigma(\alpha).$$

a) Prove that $N(\alpha) \in F$.
b) Prove that $N(\alpha\beta) = N(\alpha)N(\beta)$ for all $\alpha, \beta \in K$.
c) Find the norm of $\alpha = a + b\sqrt{d}$ for the quadratic extension $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$.

9. Let K/F be a Galois quadratic extension, $\alpha \in K$. Prove that the following two conditions are equivalent:
(1) $N(\alpha) = 1$.
(2) There exists $\beta \in K$ such that

$$\alpha = \frac{\beta}{\sigma(\beta)}$$

where σ is the nontrivial element in $Gal(K/F)$.
(Hint: Consider $\beta = 1 + \alpha$.)

10. Let p be a prime integer. Let F be a field having no nontrivial extensions of degree prime to p. Prove that every separable extension K/F of degree p is Galois.