HOMEWORK 4

1. Prove that $X^{p-1} - 1 = (X-1)(X-2) \cdot \dots \cdot (X-p+1)$ over $\mathbb{F}_p$. Derive Wilson's theorem: $(p-1)! \equiv -1$ $(mod\ p)$.

2. Prove that $f(X)^p = f(X^p)$ for every polynomial $f \in \mathbb{F}_p[X]$.

3. Let E/F be a field extension of a field F of characteristic $p > 0$. Prove that for every $\alpha \in E$ separable over F, we have $F(\alpha) = F(\alpha^p)$.

4. Prove that a field F is perfect if and only if every finite extension of F is separable.

5. Prove that every finite extension of a perfect field is also perfect.

6. For a prime p and nonzero $a \in \mathbb{F}_p$, prove that $X^p - X + a$ is irreducible and separable over $\mathbb{F}_p$. (Hint: observe that if α is a root of $X^p - X + a$ then $\alpha + 1$ is also a root.)

7. Let F be a field of characteristic p. Prove that for every $a \in F \setminus F^p$ the polynomial $X^p - a$ is irreducible over F.

8. a) Let E/F be a finite field extension of a field F of characteristic $p > 0$. An element $a \in E$ is called *purely inseparable over* F if $a^{p^n} \in F$ for some $n \geq 0$. Show that if $a \in E$ is separable and purely inseparable over F, then $a \in F$.
b) A finite extension E/F is called *purely inseparable* if all elements in E are purely inseparable over F. Show that if E/F is a finite field extension, then E is purely inseparable over the maximal separable subextension in E/F.

9. Prove that a finite field extension E/F is purely inseparable if and only if every field homomorphism $\sigma : F \to L$ has at most one extension $E \to L$.

10. Let F be a field of characteristic $p > 0$. Show that there are infinitely many fields K such that $F(X^p, Y^p) \subset K \subset F(X, Y)$. (Hint: Consider fields $F(X + f \cdot Y)$ for different polynomials $f \in F[X^p, Y^p]$.)