HOMEWORK 1

1. Prove that the subring
$$F = \{a + b\sqrt{2}; a, b \in \mathbb{Q}\} \subset \mathbb{R}$$
is a field.

2. Find the characteristic of F and the degree $[F : \mathbb{Q}]$, where F is a field in Problem 1.

3. Define the characteristic of an arbitrary commutative ring. Show that any non-negative integer can be the characteristic of some ring.

4. Show that $f = X^3+9X+6$ is irreducible in $\mathbb{Q}[X]$. Let α be a real root of f. Compute $(1+\alpha)^{-1}$ as a linear combination of $1, \alpha, \alpha^2$ with rational coefficients.

5. Show that $f = X^3 + X + 1$ is irreducible in $F = (\mathbb{Z}/2\mathbb{Z})[X]$. Let α be a root of f in some field extension of F. Compute all powers α^n, $n \geq 0$, as linear combinations of $1, \alpha, \alpha^2$.

6. Let F be a finite field. Prove that the characteristic p of F is positive. Prove that $|F| = p^n$ for some positive integer n.

7. Determine the minimal polynomial over $\mathbb{Q}$ of $1 + i$.

8. Determine the minimal polynomial over $\mathbb{Q}$ of $\sqrt{2} + \sqrt{3}$.

9. Let X be a variable over a field F, $Y = \frac{X^2}{X-1}$. Find $[F(X) : F(Y)]$.

10. Let K/F be a field extension, $\alpha, \beta \in K$. Prove that the extension $F(\alpha, \beta)/F(\alpha + \beta, \alpha\beta)$ is algebraic.