HOMEWORK 6

1. Prove that a normal closure of a separable field extension is also separable (and hence Galois).

2. Let F be the finite field $\mathbb{F}_q$ and let n be prime to q. Show that the degree of the cyclotomic extension $F(\xi_n)/F$ is equal to the order of $q + n\mathbb{Z}$ in $(\mathbb{Z}/n\mathbb{Z})^\times$.

3. Find the cyclotomic polynomial Φ_{24}.

4. Determine all roots of unity in $\mathbb{Q}(\sqrt{-3})$.

5. Find all $n \leq 10$ such that $\mathbb{Q}(\xi_n)/\mathbb{Q}$ is a cyclic extension.

6. Find the Galois group of $X^4 - a$ over $\mathbb{Q}$ if
a) $a = 1$;
b) $a = -1$;
c) $a = 2$;
d) $a = -2$.

7. Find a splitting field L of $X^4 - 4X^2 - 1$ over $\mathbb{Q}$ and describe all subfields of L.

8. Let p be a prime integer and let F be a filed of characteristic different from p and containing a primitive p-th root of unity, $a \in F^\times$. Prove that $X^p - a$ is either irreducible or split over F.

9. Let p be a prime integer and let F be a field which has no nontrivial field extensions of degree not divisible by p. Show that for any finite separable field extension L/F there is a tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_n = L$ such that for any $i = 0, 1, \ldots, n-1$, F_{i+1}/F_i is a cyclic extension of degree p.

10. Let E/F be a Galois extension and let L and K be two intermediate fields such that L/F is normal. Assume that $E = LK$ and $L \cap K = F$. Show that $Gal(E/F)$ is the semidirect product $Gal(E/L) \rtimes Gal(E/K)$.