HOMEWORK 3

1. Find an isomorphism between two finite fields $\mathbb{F}_2[X]/(X^3+X+1)\mathbb{F}_2[X]$ and $\mathbb{F}_2[X]/(X^3+X^2+1)\mathbb{F}_2[X]$.

2. Prove that $\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}$ is a normal extension.

3. Find a normal closure of the extension $\mathbb{Q}(2^{1/4})/\mathbb{Q}$ and its degree over $\mathbb{Q}$.

4. Find a normal closure of the extension $\mathbb{Q}(3^{1/3}, 3^{1/4})/\mathbb{Q}$ and its degree over $\mathbb{Q}$.

5. Let $f \in \mathbb{Q}[X]$ be an irreducible polynomial of degree 3. Assume that f has a unique real root α. Show that the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$ is not normal.

6. Let F be a field of characteristic p. Show that the rational function field $F(X)$ is a normal extension of $F(X^p)$.

7. Let f be a nonzero polynomial over a field F and g be the greatest common divisor of f and its derivative f'. Prove that the polynomial f/g is separable.

8. Let F be a field of characteristic $p > 0$ and E/F an extension of degree relatively prime to p. Show that E/F is a separable extension.

9. a) Let E/F be a finite field extension. Show that the set of all elements in E, which are separable over F, is a subfield in E containing F. It is called the *maximal separable subextension in* E/F.
b) Let F be a field of characteristic 2. Find the maximal separable subextension in $F(X)/F(X^4+X^2)$.

10. Let F be a field of characteristic p which is not a perfect field (i.e. $F^p \neq F$). Show that F has a finite extension which is not separable.