HOMEWORK 9

1. Let G be a finite abelian group such that there is a prime integer p satisfying $pa = 0$ for all $a \in G$. Prove that G is isomorphic to the direct sum of several copies of $\mathbb{Z}/p\mathbb{Z}$.

2. Determine all isomorphism classes of abelian groups of order 60.

3. Prove that every finite abelian group is isomorphic to $\mathbb{Z}/a_1\mathbb{Z} \oplus \mathbb{Z}/a_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_r\mathbb{Z}$ with a_1 dividing a_2, a_2 dividing a_3, ..., a_{r-1} dividing a_r.

4. Prove that the direct sum of n copies of $\mathbb{Z}/p\mathbb{Z}$, p is prime, cannot be generated by $n-1$ elements.

5. Prove that the symmetric group S_n is a semidirect product of A_n and $\mathbb{Z}/2\mathbb{Z}$.

6. Prove that the dihedral group D_{12} and the alternating group A_4 are not isomorphic.

7. Prove that every group of order 84 contains a normal subgroup of order 7. (Hint: Count Sylow 7-subgroups.)

8. Prove that every group of order 35 is cyclic.

9. Let a group G be a semidirect product of its subgroups K and N with N normal in G. Prove that the composition of the inclusion homomorphism $K \to G$ with the canonical homomorphism $G \to G/N$ is an isomorphism.

10. Let K and N be two subgroups of a group G with N normal in G such that the composition of the inclusion homomorphism $K \to G$ with the canonical homomorphism $G \to G/N$ is an isomorphism. Prove that G is a semidirect product of K and N.