

HOMEWORK 7

1. Let S be a subset of a group G such that for every $a \in G$ and every $s \in S$ we have $asa^{-1} \in S$. Prove that the subgroup $\langle S \rangle$ is normal in G .
2. Let N be a normal subgroup of a group G and $S \subset G$ a subset. Suppose that G is generated by the union of the two subsets N and S . Let T be the image of S under the canonical homomorphism $G \rightarrow G/N$. Prove that G/N is generated by T .
3. Let G act on a set X . Prove that if $x, y \in X$ satisfy $ax = y$ for some $a \in G$ then $G_y = a \cdot G_x \cdot a^{-1}$.
4. Let G be a group, $a \in G$. Show that the number of elements in the conjugacy class $\{bab^{-1}, b \in G\}$ divides $|G|$.
5. Let K and H be two subgroups of a group G such that $KH = G$. Prove that K acts transitively by left translations on the set of left cosets of H in G .
6. Suppose a group G acts transitively on a set of n elements. Prove that there is a subgroup of G of index n .
7. An action of a group G on a set X is called *double transitive* if for any two pairs (x_1, x_2) and (y_1, y_2) of elements of X such that $x_1 \neq x_2$ and $y_1 \neq y_2$ there is $a \in G$ such that $ax_1 = y_1$ and $ax_2 = y_2$. Prove that $|G| \geq |X|^2 - |X|$. (Hint: Consider an action of G on $X \times X$.)
8. Let H be a subgroup of a finite group G . Prove that the number of different conjugate subgroups xHx^{-1} , $x \in G$, is at most $[G : H]$.
9. Let H be a subgroup of a finite group G . Prove that if G is the union of xHx^{-1} over all $x \in G$, then $H = G$. (Hint: Use Problem 8.)
10. Let a finite group G act transitively on a set X consisting of at least two elements. Prove that there exists a $g \in G$ fixing no element of X . (Hint: Use Problem 9.)