

## HOMEWORK 4

1. Prove that for any  $n \in \mathbb{N}$  there exists a unique cyclic subgroup  $H_n \subset \mathbb{Q}/\mathbb{Z}$  of order  $n$ .
2. Let  $H_n \subset G = \mathbb{Q}/\mathbb{Z}$  be a cyclic subgroup of order  $n$ . Prove that  $G/H_n$  is isomorphic to  $G$ . (Hint: Consider the homomorphism  $f : G \rightarrow G$ ,  $f(x) = nx$ .)
3. Find all elements of finite order in  $\mathbb{R}/\mathbb{Z}$ .
4. Show that  $\text{Aut}(\mathbb{Z})$  is a cyclic group of order 2. Determine  $\text{Inn}(\mathbb{Z})$ .
5. Prove that for a group  $G$ , the subgroup  $\text{Inn}(G)$  in  $\text{Aut}(G)$  is normal.
6. Prove that  $\text{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$  is isomorphic to the symmetric group  $S_3$ . (Hint: Notice that every automorphism of  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  permutes all three nonzero elements of the group.)
7. A subgroup  $H$  in  $G$  is called *characteristic*, if  $f(H) = H$  for any automorphism  $f$  of  $G$ . Show that a characteristic subgroup  $H$  is normal in  $G$ .
8. Prove that if  $K$  is a characteristic subgroup in  $H$  and  $H$  is a characteristic subgroup in  $G$ , then  $K$  is characteristic in  $G$ .
9. Prove that if  $K$  is a characteristic subgroup in  $H$  and  $H$  is normal in  $G$ , then  $K$  is normal in  $G$ .
10. Let  $N$  be an abelian normal subgroup of a finite group  $G$ . Assume that the orders  $|G/N|$  and  $|\text{Aut}(N)|$  are relatively prime. Prove that  $N$  is contained in the center of  $G$ . (Hint: Consider the homomorphism  $f : G \rightarrow \text{Aut}(N)$ ,  $f(g)(n) = gng^{-1}$ .)