

HOMEWORK 3

1. Let H be a subgroup of a group G . Prove that if a coset aH is a subgroup of G , then $aH = H$.
2. Determine all cyclic subgroups of order 2 in $\mathbb{Z}/6\mathbb{Z}$ and their cosets.
3. Let H be the subgroup of all $n \times n$ matrices A in $G = GL_n(\mathbb{R})$ such that $\det(A) = 1$. Prove that the set of all $n \times n$ matrices B in G such that $\det(B) = 2$ is a left and right coset of H in G .
4. Let $H \subset G$ be a subgroup. Show that there is a homomorphism $f : K \rightarrow G$ from some group K to G with the image H .
5. Let $N \subset G$ be a normal subgroup. Show that there is a homomorphism $f : G \rightarrow S$ from G to some group S with the kernel N .
6. Let $H \subset G$ be a subgroup. Show that the correspondence $Ha \mapsto a^{-1}H$ is a bijection between the sets of right and left cosets of H in G .
7. Let n be a natural number. Find $Ker(f)$ and $Im(f)$ of the homomorphism
$$f : \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}, \quad f(a + \mathbb{Z}) = na + \mathbb{Z}.$$
8. Let $K \subset H \subset G$ be subgroups. Show that if K has finite index in G then $[G : K] = [G : H][H : K]$.
9. Let $H \subset G$ be a subgroup. Suppose that for any $a \in G$ there exists $b \in G$ such that $aH = Hb$. Show that H is normal in G .
10. Show that every subgroup of index 2 is normal.