HOMEWORK 2

1. Prove that the group $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ is not cyclic.

2. Prove that if a is an element of a finite group G such that $ord(a) = |G|$, then G is cyclic.

3. Find a non-cyclic group of order 4.

4. Let $f : G \to H$ be an isomorphism. Prove that for every $a \in G$, the elements a and $f(a)$ have the same order.

5. Let K and H be two subgroups of a group G. Prove that the union $K \cup H$ is a subgroup if and only if either $K \subset H$ or $H \subset K$.

6. Show that if K and H are two finite subgroups in G of relatively prime order (i.e., $|K|$ and $|H|$ have no common divisor greater than 1), then $K \cap H = \{e\}$.

7. Find all subgroups of the symmetric group S_3.

8. Let G be a group and $a \in G$. Prove that there is a homomorphism $f : \mathbb{Z} \to G$ such that $f(1) = a$. Show that f is unique.

9. Prove that every homomorphism $f : \mathbb{Q} \to \mathbb{Z}/m\mathbb{Z}$ is trivial, i.e. $f(x) = [0]_m$ for all $x \in \mathbb{Q}$.

10. Show that if a group G has only finite number of subgroups, then G is finite. (Hint: Note that G is a union of cyclic subgroups.)