Problem Set 3
Due Friday, October 20.

Foundations of Number Theory

Math 435, Fall 2006

1. Show that no numbers of the form p^α (p prime, $\alpha \in \mathbb{N}$) and pq (p, q distinct prime numbers, $pq \neq 6$) are perfect.
2. Show that if $n \in \mathbb{N}^{>0}$ is perfect, then
$$\sum_{d|n} \frac{1}{d} = 2.$$
3. Let 0 denote the number-theoretic function given by $0(n) = 0$ for all $n \in \mathbb{N}^{>0}$. Show that if f, g are number-theoretic functions with $f * g = 0$, then $f = 0$ or $g = 0$.
4. Show: if $\sigma(n)$ is odd, then n is a square or twice a square.
5. Let f be a number-theoretic function.
   (a) Show that there exists a number-theoretic function g with $f * g = \varepsilon$ if and only if $f(1) \neq 0$. (Recall that ε denotes the number-theoretic function with $\varepsilon(1) = 1$ and $\varepsilon(n) = 0$ for $n > 1$.)
   (b) Show that if g, h are number-theoretic functions with $f * g = \varepsilon$ and $f * h = \varepsilon$, then $g = h$.

   If $f(1) \neq 0$, then we denote the unique number-theoretic function g with $f * g = \varepsilon$ by f^{-1}.
6. Let f be a multiplicative number-theoretic function with $f(1) \neq 0$. Show that the following are equivalent:
   (a) f is completely multiplicative;
   (b) $f^{-1} = \mu f$ (pointwise multiplication);
   (c) $f^{-1}(p^\alpha) = 0$ for all prime numbers p and $\alpha \in \mathbb{N}$, $\alpha \geq 2$.
7. (Extra credit.) Show that $\tau^3 * 1 = (\tau * 1)^2$.